Prof. Vincent Tassion HS 2024

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 1

Exercise 1. This exercise shows that a simple random walk on $\mathbb{Z}$ cannot be confined to a strip forever. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables defined by

$$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = -1) = 1/2.$$

For $n \geq 1$, define $S_n = X_1 + \cdots + X_n$. Let $k \geq 1$ be a fixed integer. Show that

$$\mathrm{P}\big(\forall n \geq 1 \;\; 0 \leq S_n \leq k\big) = 0.$$

Exercise 2 [R]. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniformly distributed in $\{-1, 1, 2, -2\}$. For $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Fix $n \geq 1$.

(i) Compute $\mathrm{E}(S_n)$ and $\mathrm{Var}(S_n)$.

(ii) Prove that

$$\mathrm{P}(|S_n| \geq 2\sqrt{n}) \leq \frac{3}{4}.$$

(iii) Prove that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_n = k) = \mathrm{P}(S_n = -k).$$

(iv) Provethat

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(X_1 + \cdots + X_n = k) = \mathrm{P}(X_{n+1} + \cdots + X_{2n} = k).$$

(v) Deduce that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) = \sum_{i \in \mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(S_n = k - i).$$

(vi) Apply the Cauchy-Schwarz inequality to show that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) \leq \mathrm{P}(S_{2n} = 0).$$

(vii) Deduce that

$$\mathrm{P}(S_{2n} = 0) \geq \frac{1}{50\sqrt{n}}.$$

Exercise 3. Let $(X_n)_{n\geq 1}$ be iid Exp(1) random variables. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \quad a.s.$$

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of events such that

$$\lim_{n\to\infty} \mathrm{P}(A_n) = 0 \text{ and } \sum_{n\geq 1} \mathrm{P}(A_n \setminus A_{n+1}) < \infty.$$

Prove that P(infinitely many A_n occur) $= 0$.

Submission of solutions. Hand in your solutions by 18:00, 27/09/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 2

Exercise 1. Give an example of a subsequence $(n(k))_{k\geq 1}$ such that

$$X_{n(k)} \xrightarrow{a.s.} 0,$$

where

(i) $(X_n)_{n\geq 1}$ is iid with $X_1 \sim \mathrm{Ber}(1/n)$.
(ii) $(X_n)_{n\geq 1}$ is the typesetter sequence.

Exercise 2 [R]. Let (E, d) and (E', d') be metric spaces. Let $(X_n)_{n\geq 1}$ and X be random variables taking values in E.

(i) (Subsubsequence lemma) Show that X_n converges to X in probability if and only if for every subsequence $(n(k))_{k\geq 1}$ there exists a subsubsequence $(n(k(l))_{l\geq 1}$ such that $X_{n(k(l))}$ converges to X almost surely as $l \to \infty$.
(ii) (Continuous mapping) Let $f : E \to E'$ be a continuous function. First, suppose $X_n \to X$ a.s. and show that $f(X_n) \to f(X)$ a.s. Next, suppose $X_n \to X$ in probability and show that $f(X_n) \to f(X)$ in probability.

Exercise 3 [R]. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables such that $Y_n \sim \mathrm{Exp}(\lambda_n)$, where $(\lambda_n)_{n\geq 1}$ is a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$.

(i) Show that $Y_n \to 0$ in probability.
(ii) Let $\lambda_n = 10 \log n$. Does Y_n converge to 0 almost surely?
(iii) Let $\lambda_n = (\log n)^2$. Does Y_n converge to 0 almost surely?

Exercise 4. Define the space of functions

$$L^0 = \{X : \Omega \to E \text{ measurable}\}/\sim,$$

where the equivalence relation $\sim$ is defined by

$$X \sim Y \iff X = Y \ a.s.$$

(i) Show that $D(X, Y) = \mathrm{E}\big(1 \wedge d(X, Y)\big)$ defines a metric on L^0.
(ii) Assume E is complete. Show that (L^0, D) is complete.

Exercise 5. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables with $\mathrm{E}(|X_1|) < \infty$. Define

$$S_n = \sum_{i=1}^{n} X_i X_{i+1}.$$

Show that S_n/n converges almost surely.

Submission of solutions. Hand in your solutions by 18:00, 04/10/2024 following the instructions on the course website

https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 3

Exercise 1. Let $(u_n)_{n\geq 1}$ and c be real numbers. Suppose $\lim_{n\to\infty} u_n = c$. Show that

$$\lim_{n\to\infty} \frac{u_1 + \cdots + u_n}{n} = c.$$

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be pairwise independent, positive, identically distributed random variables with $\mathrm{E}(X_1) = \infty$. Show that

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \infty.$$

Hint: for $a > 0$ consider the random variables $\min(X_n, a)$.

Exercise 3. [Hard] Give an example of an iid sequence $(X_n)_{n\geq 1}$ such that almost surely

$$\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = -\infty.$$

Exercise 4. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables that are uniformly distributed in unit ball $\{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$. Define $(Z_n)_{n\geq 1}$ inductively by $Z_0 = (1,0)$ and $Z_{n+1} = \|X_{n+1}\|_2 \cdot Z_n$.

(i) Show that there exists $c \in \mathbb{R}$ such that

$$\frac{\log \|Z_n\|_2}{n} \xrightarrow[n\to\infty]{a.s.} c.$$

(ii) Compute the value of c.

(iii) What is the limit when $Z_0 = (2,2)$?

Exercise 5. [R]

(i) Show that a family of random variables $(X_i)_{i\in I}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Show this family is uniformly integrable if and only if it is bounded in L^1 (that is, there exists $M \in \mathbb{R}$ such that for all $i \in I$, $\mathrm{E}(|X_i|) \leq M$) and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq \delta$ and all $i \in I$ we have $\mathrm{E}(|X_i| 1_A) \leq \epsilon$.

(ii) Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be two uniformly integrable families of random variables. Show that $(X_i + Y_j)_{(i,j)\in I\times J}$ is uniformly integrable.

Submission of solutions. Hand in your solutions by 18:00, 11/10/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 4

Exercise 1. [R] Let $(X_n)_{n\geq 1}, X$ be random variables such that $X_n \xrightarrow{\mathrm{P}} X$ as $n \to \infty$. Show that the following are equivalent.

(1) $(X_n)_{n\geq 1}$ is uniformly integrable.
(2) $(X_n), X$ are all in L^1 and $\mathrm{E}[|X_n|] \to \mathrm{E}[|X|]$ as $n \to \infty$.

Exercise 2. [R] Give an example of a sequence of random variables $(X_n)_{n\geq 1}$ that is not uniformly integrable and a random variable X such that

$$X_n \xrightarrow{\mathrm{P}} X \quad \text{and} \quad \mathrm{E}(X_n) \to \mathrm{E}(X),$$

as $n \to \infty$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of iid real-valued random variables. Show that if $\mathrm{E}(|X1|) < \infty$, then the sequence $(\max(X_1, \ldots, X_n)/n)_{n\geq 2}$ is uniformly integrable. Is the converse true?

Exercise 4. Consider the probability space defined by $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^\Omega$, and

$$\forall A \in \mathcal{F} \quad \mathrm{P}(A) = |A|/6.$$

Define random variables X and Y by

$$X(\omega) = \omega \pmod 2 \quad \text{and} \quad Y(\omega) = \omega \pmod 3.$$

Is $\sigma(X, Y) = \sigma(X) \cup \sigma(Y)$?

Exercise 5. [R] Let $p \in [0, 1]$. Let $(X_n)_{n\geq 1}, Y$ be independent random variables with distributions specified as follows: $(X_n)_{n\geq 1}$ is an iid sequence of random variables with $\mathrm{P}(X_1 = 1) = 1 - \mathrm{P}(X_1 = -1) = p$ and $\mathrm{P}(Y = 1) = \mathrm{P}(Y = -1) = 1/2$. For $n \geq 1$, define $Z_n = X_n \cdot Y$. For which values of $p \in [0, 1]$ is the tail sigma algebra of $(Z_n)_{n\geq 1}$ trivial?

Submission of solutions. Hand in your solutions by 18:00, 18/10/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 5

Exercise 1. Let $\alpha, \beta > 0$ be real numbers. Let $X \sim \mathrm{Poi}(\alpha)$ and $Y \sim \mathrm{Poi}(\beta)$ be independent random variables. Show that $X + Y \sim \mathrm{Poi}(\alpha + \beta)$.

Exericse 2. [R] This exercise shows that the tail of a random variable is determined by the behaviour of its characteristic function around zero. Let X be a real-valued random variable and let ϕ be its characterisitic function. Show that

$$\mathrm{P}(|X| > 2/u) \leq \frac{1}{u} \int_{-u}^{u} \big(1 - \phi(t)\big)\, dt.$$

Exercise 3. [R] Let X be a real-valued random variable such that its characteristic function $\phi_X \in L^1(\mathbb{R})$.

(i) Show that for all

$$\forall \psi \in \mathcal{C}_c^\infty \quad \mathrm{E}\big(\psi(X)\big) = \int_{\mathbb{R}} \psi(x) \int_{\mathbb{R}} \phi(t) e^{-itx}\, dt\, dx.$$

(ii) Deduce that X has a density.

Exercise 4. Let $X_0, X_1, \ldots$ be iid random variables with

$$\mathrm{P}(X_0 = 1) = \mathrm{P}(X_0 = -1) = 1/2.$$

For $n \geq 1$ define

$$Y_n = X_0 \cdots X_n.$$

Let

$$\mathcal{X} = \sigma(X_1, X_2, \ldots) \quad \text{and} \quad \mathcal{Y}_n = \sigma(Y_n, Y_{n+1}, \ldots).$$

The aim of this exercise is to show that

$$\bigcap_{n\geq 1} \sigma(\mathcal{X}, Y_n) \quad \text{and} \quad \sigma\Big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\Big)$$

are not equal.

(i) Show that $\sigma(X_0) \subset \sigma(\mathcal{X}, Y_n)$ for each $n \geq 1$.

(ii) Show that $\bigcap_{n\geq 1} \mathcal{Y}_n$ is trivial.

(iii) Show that $\sigma(X_0)$ is independent of $\sigma\big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\big)$. (Hint: check independence on a suitable π-system.)

(iv) Conclude.

Submission of solutions. Hand in your solutions by 18:00, 25/10/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 6

Exercise 1. Let $(X_n)_{n\geq 1}$ be iid random variables in L^2 such that X_1 has the same law as $-X_1$, $\mathrm{P}(X_1 = 0) > 0$, and $X_1 \in \mathbb{Z}$ a.s. For $n \geq 1$, define

$$S_n = X_1 + \cdots + X_n.$$

Show that there exists $c > 0$ such that

$$\mathrm{P}(S_n = 0) \underset{n\to\infty}{\sim} \frac{c}{\sqrt{n}}.$$

(For two sequences (a_n) and (b_n) of real numbers, we write $a_n \underset{n\to\infty}{\sim} b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.)

Exercise 2. Let $(X_n)_{n\geq 1}$ be a an iid sequence of random variables with

$$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = -1) = 1/2.$$

(i) Show that there exists a constant $c > 0$ such that for all $n \geq 1$ and positive real numbers $a_1, \dots, a_n > 0$ we have

$$\mathrm{P}(a_1 X_1 + \cdots + a_n X_n = 0) \leq \frac{c}{\sqrt{n}}.$$

(ii) Show that there exists a constant $c > 0$ such that for all $n \geq 1$ we have

$$\mathrm{P}(X_1 + 2X_2 + \cdots + nX_n = 0) \leq \frac{c}{n^{3/2}}.$$

Exercise 3 (The moment problem). [R]

In this exercise, we only consider random variables that are in L^p for all $p \geq 1$. We say that X is determined by its moments if for all random variables Y such that

$$\forall n \geq 1 \quad \mathrm{E}(X^n) = \mathrm{E}(Y^n), \tag{1}$$

we have $\mu_X = \mu_Y$.

(i) We first give an example of a random variable that is not determined by its moments. Let

$$X \sim e^Z \quad \text{where } Z \sim \mathcal{N}(0,1).$$

Let Y be a random variable taking values in $\{e^k : k \in \mathbb{Z}\}$ defined as follows:

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(Y = e^k) = \frac{e^{-k^2/2}}{\Lambda} \quad \text{where } \Lambda = \sum_{k\in\mathbb{Z}} e^{-k^2/2}.$$

Show that

$$\forall n \geq 1 \quad \mathrm{E}(X^n) = \mathrm{E}(Y^n) = e^{n^2/2}.$$

(ii) Let X be a random variable such that there exists $t > 0$ such that $\mathrm{E}\big(e^{t|X|}\big) < \infty$. We show that then X is determined by its moments. First, check that $X \in L^p$ for all $p \geq 1$ and ϕ_X, the characteristic function of X, is infinitely differentiable on $\mathbb{R}$.

(iii) Fix $a \in \mathbb{R}$. Show that

$$\forall \epsilon \in (-t, t) \quad \phi_X(a + \epsilon) = \sum_{k=0}^{\infty} \frac{e^k}{k!} \phi_X^{(k)}(a).$$

(iv) Let ϕ_Y be the characteristic function of Y. Show that

$$\forall \epsilon \in (-t, t) \quad \phi_X(\epsilon) = \phi_Y(\epsilon).$$

(v) Show that $\phi_X(\epsilon) = \phi_Y(\epsilon)$ for all $\epsilon \in \mathbb{R}$. Conclude that $\mu_X = \mu_Y$.

Submission of solutions. Hand in your solutions by 18:00, 2/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 7

Exercise 1. Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be a continuous function. Let $(X_n)_{n\geq 1}, X$ be random variables taking values in E such that

$$X_n \xrightarrow{(d)} X.$$

Show that

$$f(X_n) \xrightarrow{(d)} f(X).$$

Exercise 2. [R] Let $p \in (0, 1)$ and let $(X_n)_{n\geq 1}$ be a sequence of random variables where $X_n \sim \mathrm{Geo}(p/n)$. Show that X_n/n converges in distribution to a random variable Y. What is the distribution of Y?

Exercise 3. [R] Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables where X_n has density p_n (with respect to Lebesgue measure Leb). Suppose there is a measurable function such that for Leb-almost all $x \in \mathbb{R}$ we have

$$p_n(x) \to p(x) \text{ as } n \to \infty.$$

(i) Is p always the density of some random variable? Justify your answer.

(ii) Assume that there is an integrable measurable function (with respect to Leb)

$$q : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

such that for all $n \geq 1$ and Leb-almost all x we have

$$p_n(x) \leq q(x).$$

Then show that p is the density of some random variable X and that X_n converges in distribution to X.

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in distribution to a uniformly distributed random variable in $[0, 1]$. Let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in probability to 0. Show that

$$\mathrm{P}(X_n < Y_n) \to 0 \text{ as } n \to \infty.$$

Submission of solutions. Hand in your solutions by 18:00, 8/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 8

Exercise 1. [R] Let $(X_n)_{n\geq 1}$ be an iid sequence of $\mathcal{N}(0,1)$ random variables. For $n \geq 1$, define

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} \sqrt{k} X_k.$$

Does Y_n converge in distribution? What is the limit?

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be a sequence of iif $\mathcal{U}[0,1]$ random variables.

(i) Show that $n\min(X_1,\dots,X_n)$ converges in distribution to a random variable Y. What is the distribution of Y?

(ii) Show that

$$(X_1 + \cdots + X_n)\min(X_1,\dots,X_n) \xrightarrow{(d)} Y/2.$$

Exercise 3 (Normality of the t-statistic). [R] Let $(X_n)_{n\geq 1}$ be iid real-valued random variables in L^2. Let $m = \mathrm{E}(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1$. For $n \geq 1$, define

$$\bar{X}_n = \frac{X_1 + \cdot + X_n}{n} \quad \text{and} \quad S_n^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2.$$

The aim of this exercise is to show that

$$\frac{X_1 + \cdots + X_n - nm}{\sqrt{nS_n^2}} \xrightarrow{(d)} Z, \text{ where } Z \sim \mathcal{N}(0,1). \tag{1}$$

(i) Show that $S_n^2 \to \sigma^2$ a.s.

(ii) Show that $\frac{X_1+\cdots+X_n}{\sqrt{n\sigma^2}} \xrightarrow{(d)} Z$, where $Z \sim \mathcal{N}(0,1)$.

(iii) Prove (1).

Exercise 4 (Skorokhod representation on the reals). Let $(X_n)_{n\geq 1}, X$ be real-valued random variables such that $X_n \xrightarrow{(d)} X$. The aim of this is to construct a probability space carrying these random variables such that $X_n \xrightarrow{a.s.} X$. For a distribution function F, we define

$$F^{-1} : (0,1) \to \mathbb{R}, \quad \text{by } F^{-1}(t) = \inf\{s : F(s) > t\}.$$

Let (F_n) and F be the distribution functions of (X_n) and (X), and let $U \sim \mathcal{U}(0,1)$.

(i) Show that $F_n^{-1}(U)$ has the same distribution as X_n for all $n \geq 1$ and that $F^{-1}(U)$ has the same distribution as X.

(ii) Show that

$$F_n^{-1}(U) \xrightarrow{a.s.} F^{-1}(U) \quad \text{as } n \to \infty.$$

Submission of solutions. Hand in your solutions by 18:00, 16/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 9

Exercise 1. [R] Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\mathcal{A} = \{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ be a partition of Ω. Let X be a real-valued $\sigma(\mathcal{A})$-measurable random variable. Show that there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$X = \sum_{i=1}^{n} \lambda_i 1_{\Omega_i}.$$

Exercise 2. [R] Fix $n \geq 1$. Let $X \sim \mathrm{Unif}[0,1]$ and let $Y = \lfloor n \cdot X \rfloor$. Compute $\mathrm{E}(X|Y)$.

Exercise 3. Fix $n \geq 2$. Let X, Y be two numbers chosen uniformly at random from $\{1, 2, \ldots, n\}$ without replacement. Define the event $A = \{Y > X\}$.

(i) Compute $\mathrm{E}(Y|A)$.
(ii) Compute $\mathrm{E}\big(\max(X,Y) | \min(X,Y)\big)$.

Exercise 4. Let X, Y be real-valued random variables taking finitely many values. Define the random variable

$$\mathrm{Var}(X|Y) = \mathrm{E}(X^2|Y) - \mathrm{E}(X|Y)^2.$$

Show that

$$\mathrm{Var}(X) = \mathrm{E}\big(\mathrm{Var}(X|Y)\big) + \mathrm{Var}\big(\mathrm{E}(X|Y)\big).$$

Exercise 5. Let $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}, X, Y$ be random variables. Assume that for all $n \geq 1$, X_n and Y_n are independent, and that X and Y are independent. Suppose

$$X_n \xrightarrow{(d)} X \quad \text{and} \quad Y_n \xrightarrow{(d)} Y.$$

Then show that

$$(X_n, Y_n) \xrightarrow{(d)} (X, Y).$$

Submission of solutions. Hand in your solutions by 18:00, 22/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 10

Exercise 1. [R] Let A be an compact set in $\mathbb{R}^2$ and let $(X, Y) \sim \mathrm{Unif}(A)$. Compute

$$\mathrm{E}\big(X^2|Y\big)$$

in the following cases:
(1) $A = [-1, 1]^2$,
(2) $A = \{(x, y) : |x| + |y| \leq 1\}$.

Exercise 2. Let X, Y be independent random variables and let $\psi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a measurable function such that

$$\mathrm{E}\big(|\psi(X, Y)|\big) < \infty.$$

Define $\phi : \mathbb{R} \to [0, \infty]$ by

$$\phi(y) = \mathrm{E}\big(\psi(X, y)\big).$$

Show that

$$\mathrm{E}(\psi(X, Y)|Y) = \phi(Y) \ a.s.$$

Exercise 3. [R] Let $(Y_n)_{n\geq 1}$ be iid random variables which are uniform in $\{-1, +1\}$ and let X be a random variable in L^2. Let $[n]$ denote $\{1, \dots, n\}$ and for a subset $S \subset [n]$, define

$$Y_S = \prod_{i \in S} Y_i,$$

where $Y_\emptyset$ defined to be 1.
(1) Show that $\mathrm{E}(X|Y_1) = \mathrm{E}(X) + \mathrm{E}(XY_1)Y_1$.
(2) More generally, for all $n \geq 1$ show that

$$\mathrm{E}(X|Y_1, \dots, Y_n) = \sum_{S \subset [n]} \mathrm{E}(XY_S)Y_S.$$

Exercise 4. [R] Let X be a real-valued random variable defined on $(\Omega, \mathcal{F}, \mathrm{P})$ that takes values in $[0, \infty]$ a.s. Let $\mathcal{G} \subset \mathcal{F}$ be a sigma-algebra. Define $\mathrm{E}(X|\mathcal{G})$ and show that it is unique (up to almost sure equivalence).

Submission of solutions. Hand in your solutions by 18:00, 29/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 11

Exercise 1. [R] Let $(X_n)_{n\geq 1}$ be iid random variables in L^1 and for $n \geq 1$, let

$$S_n = X_1 + \cdots + X_n.$$

Compute $\mathrm{E}(S_n|X_1)$ and $\mathrm{E}(X_1|S_n)$.

Exercise 2. [R] Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be sigma-algebras and let X be a random variable. Show that we need not have that

$$\mathrm{E}(\mathrm{E}(X|\mathcal{G})|\mathcal{H}) = \mathrm{E}(X|\mathcal{G} \cap \mathcal{H}).$$

Exercise 3. [R] Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{+1. -1\}$ with $\mathrm{P}(X_1 = 1) = 1/2$. Let $S_0 = 0$ and for $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geq 1$, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$M_n = S_n^2 - n$$

is a $(\mathcal{F}_n)$-martingale.

Exercise 4. Fix $p \in (0, 1)$. Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{+1, -1\}$ with $\mathrm{P}(X_1 = 1) = p$. Let $S_0 = 0$ and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geq 1$, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}$$

is a $(\mathcal{F}_n)$-martingale.

Exercise 5 (Azuma's inequality). [R] Let $(X_n)_{n\geq 0}$ be martingale with respect to its canonical filtration $(\mathcal{F}_n)_{\geq 0}$. Assume $X_0 = 0$ and that $|X_n - X_{n-1}| \leq 1$ for all $n \geq 1$. Fix $m \geq 1$. The aim of this exercise is to show that $\lambda > 0$ we have

$$\mathrm{P}(X_m > \lambda\sqrt{m}) \leq e^{-\lambda^2/2}. \quad (1)$$

(1) Let $\alpha > 0$. Show that for all $x \in [-1, 1]$ we have $e^{\alpha x} \leq \frac{e^{\alpha}+e^{-\alpha}}{2} + \frac{e^{\alpha}-e^{-\alpha}}{2}x$
(2) Set $Y_i = X_i - X_{i-1}$. Show that for all $i \geq 1$ we have

$$\mathrm{E}(e^{\alpha Y_i}|\mathcal{F}_{i-1}) \leq \cosh(\alpha) \leq e^{\alpha^2/2}.$$

(3) Deduce that $\mathrm{E}(e^{\alpha X_m}) \leq e^{\alpha^2 m/2}$.
(4) Use $\alpha = \lambda/\sqrt{m}$ and Markov's inequality to prove (1).

Submission of solutions. Hand in your solutions by 18:00, 06/12/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 12

Exercise 1. [R]

(1) Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniform in $\{-1, 1\}$. Show that

$$S_n = \sum_{m=1}^{n} \frac{X_m}{m^{3/4}}$$

converges almost surely as $n \to \infty$.

(2) Find an example of a martingale that converges almost surely but is not bounded in L^1.

(3) Find an example of a martingale that converges almost surely to ∞.

Exercise 2. Let $(Y_n)_{n\geq 0}$ be a sequence of non-negative iid random variables with $\mathrm{E}(Y_1) = 1$ and $\mathrm{P}(Y_1 = 1) < 1$ and let $(\mathcal{F}_n)_{n\geq 0}$ be the canonical filtration.

(1) Show that $X_n = \prod_{k=0}^{n} Y_k$ defines a martingale with respect to $(\mathcal{F}_n)$.

(2) Show that $X_n \to 0$ as $n \to \infty$ a.s.

Exercise 3. Fix $p \in (0, 1/2)$. Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{-1, 1\}$ with $\mathrm{P}(X_1 = 1) = p$. For $n \geq 1$ let $S_n = X_1 + \cdots + X_n$ and let

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

Show that M_n converges almost surely to 0 but $\mathrm{E}(M_n)$ does not converge to 0 as $n \to \infty$.

Exercise 4 (Positive harmonic functions on the square lattice). Let

$$h : \mathbb{Z}^2 \to \mathbb{R}_{>0}$$

be a harmonic function, meaning that

$$\forall (x, y) \in \mathbb{Z}^2 \quad h(x, y) = \frac{1}{4}\big(h(x+1, y) + h(x-1, y) + h(x, y+1) + h(x, y-1)\big).$$

The aim of this exercise is to show that h must be constant. Let $(X_n)_{n\geq 1}$ be iid uniform in $\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Define the sequence $(Z_n)_{n\geq 0}$ by $Z_0 = (0, 0)$ and

$$Z_n = \sum_{k=1}^{n} X_k$$

for $n \geq 1$. Let $(\mathcal{F}_n)$ be the filtration generated by (Z_n).

(1) Show that $\big(h(Z_n)\big)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converges almost surely.

(2) You may use the fact that

$$\forall (x, y) \in \mathbb{Z}^2 \quad |\{n : Z_n = (x, y)\}| = \infty \quad a.s.$$

Conclude that h is consant.

(3) Instead of assuming h takes positive values, assume that $|h|$ is bounded. Then show that h is constant.

Exercise 5 (Pólya's Urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$ a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. We let $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$.

(1) Prove that B_n is uniformly distributed on $\{0, 1, \ldots, n\}$.
(2) Let $M_n = (B_n + 1)/(n+2)$ be the proportion of black balls in the urn just after time n. Prove that (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and show that $M_n \to U$ as $n \to \infty$ a.s. for some random variable U.
(3) Show that U is uniformly distributed on $(0, 1)$.

Submission of solutions. Hand in your solutions by 18:00, 13/12/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 13

Exercise 1. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and let S, T be two stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or disprove with a counter-example the following statements:
(1) $S \vee T$ is a stopping time.
(2) $S \wedge T$ is a stopping time.
(3) $S + T$ is a stopping time.
(4) $S + 1$ is a stopping time.
(5) $S - 1$ is a stopping time.

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be iid random variables uniform in $\{-1, 1\}$. Let $S_0 = 0$ and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$. Fix integers $a < 0 < b$. For an integer k, define $T_k = \min\{n \geq 0 : S_n = a\}$. Define

$$T_{a,b} = T_a \wedge T_b.$$

(1) Show that $T_{a,b}$ is a stopping time that is finite almost surely.
(2) Compute $\mathrm{P}(T_a < T_b)$.
(3) Compute $\mathrm{E}(T_{a,b})$.

Exercise 3. [R] Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n\geq 0}$ stopping time.
(1) Assume that $\mathrm{E}(T) < \infty$ and there there exists $K > 0$ such that a.s. we have

$$\mathrm{E}(|M_{n+1} - M_n|) \mid \mathcal{F}_n \leq K$$

for every $n \geq 0$. Show that $\mathrm{E}(M_T) = \mathrm{E}(M_0)$.
Hint. Justify that $|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| 1_{T>i}$ and use dominated convergence.
(2) Let $(X_n)_{n\geq 1}$ be iid L^1 real-valued random variables. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(S_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a $(\mathcal{F}_n)$-stopping time with $\mathrm{E}(T) < \infty$. Show that

$$\mathrm{E}(S_T) = \mathrm{E}(X_1)\mathrm{E}(T).$$

Exercise 4. Let $(M_n)_{n\geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$.
(1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n\geq 0}\}$ is uniformly integrable?
(2) Let T be a stopping time. Is it true that $(M_{n\wedge T})_{n\geq 0}$ is a uniformly integrable martingale? Justify your answer.

Submission of solutions. Hand in your solutions by 18:00, 20/12/2024 following the instructions on the course website

https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/

Note that only the exercises marked with [R] will be corrected.