Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 2

Jörg Nick, Thea Kosche

October 17, 2024

Exercise 1. *Let* $\Omega = B(0,1) \subset \mathbb{R}^d$ *be the open ball of radius 1 centered at* $0 \in \mathbb{R}^d$. *Find* $\phi \in H^1(\Omega)$ *such that* $\Delta\phi \in L^2(\Omega)$ *however* $\phi \notin H^2(\Omega)$. *In other words find* $\phi \in H^1(\Omega, \Delta)$ *such that* $\phi \notin H^2(\Omega)$.

Exercise 2. *Let* $\Omega = \{x \in \mathbb{R}^2 | x_1, x_2 \in (-1,1)\}$ *denote the open cube centered at the origin with length* 2. *Show that the trace operator* $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ *fulfills*

$$\|\gamma_0\|_{L^2(\partial\Omega)\leftarrow H^1(\Omega)} \leq 2.$$

You can find some hints on the next page.

Hint 1: Show the estimate for $\gamma : C^\infty(\overline{\Omega}) \cap H^1(\Omega) \to L^2(\partial\Omega)$ and argue by density.
Hint 2: Show the estimates only for one of the four edges of $\partial\Omega$ and use the symmetry.
Hint 3: Use the auxiliary function $s \mapsto su(s,1)^2$ for $s \in (0,1)$ and make use of the fundamental theorem of calculus.