Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 3

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Exercise 1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain with connected boundary Γ. Consider the Helmhomltz equation with Neumann boundary condition*

$$-\Delta u + u = f \quad \textit{in } \Omega, \quad \frac{\partial u}{\partial n} = g \quad \textit{on } \Gamma \tag{1}$$

Show that for $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, the following conditions are equivalent:

1. *u is solution of* (1)*.*
2. *It holds*
$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + uv \right) d(x,y) = \int_\Omega f v \cdot d(x,y) + \int_\Gamma g v d\sigma$$
for all $v \in C^1(\Omega) \cap C(\bar{\Omega})$.
3. *u minimizes*
$$\frac{1}{2}\int_\Omega \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + v^2 \right] d(x,y) - \int_\Omega f v d(x,y) - \int_\Gamma g v d\sigma = \min!$$
among all $v \in C^1(\Omega) \cap C(\bar{\Omega})$.