Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 4

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Exercise 1. *Let X be a Banach space. Let $a : X \times X \to \mathbb{R}$ be a bilinear form and let $l : X \to \mathbb{R}$ be a linear functional on X.*
Let $Y \subset X$ be a d-dimensional subspace. Consider the following variational problem on Y: Find $u_Y \in Y$, such that

$$a(u_Y, v) = l(v) \quad \text{for all } v \in Y.$$

Suppose, you would want to implement this variational problem on your computer. What would you do?

1. *Reformulate this formulation as a linear system of equations, i.e. find $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$ such that the solution $x \in \mathbb{R}^d$ of*

$$Ax = b, \tag{1}$$

   uniquely determines the solution u_Y.

2. *What does the bilinear form $a(u, v)$ need to satisfy such that the matrix A is invertible?*

Exercise 2. *Given the following boundary value problem on $H^1((0, 1))$*

$$\begin{aligned} -u''(x) &= \pi^2 \sin(\pi x), \quad x \in (0, 1), \\ u(0) &= u(1) = 0. \end{aligned} \tag{2}$$

In this exercise you are going to apply what you derived in Exercise 1 to solve this boundary value problem numerically.

1. *Reformulate the boundary value (2) as a variational problem on $H^1_0((0, 1))$.*
2. *Find a sequence of finite dimensional vector spaces $Y_n \subset H^1_0((0, 1))$, such that $Y_n \subsetneq Y_{n+1}$ and implement the corresponding linear system (1).*[1]
3. *Determine your rate of convergence for increasing n theoretically.*[2]
4. *Verify your predicted, theoretical rate of convergence numerically.*

If you get stuck, you can use the template provided on website.

[1] In order to determine A feel free to use symbolic integral solvers, such as Wolfram alpha. For the right hand side b, you should use a quadrature algorithm.

[2] For this it might be useful to guess the solution of (2).