Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 5

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Exercise 1. *Let* $\mathcal{G} := \{\tau_i : 0 \leqslant i \leqslant n\}$ *be a partition of* $\Omega = [a, b]$*, i.e.* $\tau_i := [x_i, x_{i+1}]$ *for some* $a = x_0 < x_1 < \ldots < x_{n+1} = b$*. Let* $S^{1,0}_{\mathcal{G},0} = \operatorname{span}\{b_i(x), 1 \leqslant i \leqslant n\}$*, where* $b_i(x)$ *are the piecewise linear "hat functions" on* $\mathcal{G}$*.*[1]

(a) *Implement a program (Matlab or Python), which, for a given partition* $\mathcal{G}$ *and a given function* $f \in C^0(\Omega)$*, computes the solution of the linear system* $\mathbf{A}\mathbf{u} = \mathbf{r}$*, where*

$$\mathbf{A} = (a_{ij})_{i,j=1}^n\,, \quad a_{ij} := \int_0^1 b_i' \cdot b_j' + \int_0^1 b_i \cdot b_j \quad \text{and} \quad \mathbf{r} = (r_i)_{i=1}^n\,, \quad r_i := \int_0^1 f \cdot b_i$$

Find an explicit formula for the entries of $\mathbf{A}$*. You can use pre-built routines to compute the vector* $\mathbf{r}$ *(e.g. the Matlab function integral).*

(b) *Use your program to determine approximated solutions of the boundary value problem*

$$-u''(x) + u(x) = 1 \quad \text{in } \Omega = (0,1), \quad u(0) = u(1) = 0$$

Plot the exact solution of the problem (BVP) and the approximated solutions for different partitions $\mathcal{G}$*.*

Exercise 2. *Let* $\Omega := (0, 1)$*, define*

$$\chi(x) := \begin{cases} \frac{1}{2}, & \text{if } x \in \left(0, \frac{1}{2}\right), \\ -\frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

(a) *Show that* $l : H^1(\Omega) \to \mathbb{R}, l(v) := \int_\Omega \chi v'$ *is continuous but has no representation of the form:*

$$l(v) = \int_\Omega f v, \text{ for some } f \in L^2(\Omega), \forall v \in H^1(\Omega)$$

(b) *Show that* $u(x) := -\frac{1}{2}\left|x - \frac{1}{2}\right| + \frac{1}{2}$ *is the solution of the variational problem* $\int_\Omega \langle \nabla u, \nabla v\rangle = l(v), \forall v \in H^1_0(\Omega)$*.*

(c) *Show that* $u \notin H^2(\Omega)$*.*

[1] In case, you used these functions to solve the exercise of last week, use the function spaces $Y_n = \operatorname{span}(Y_{n-1}, x \mapsto x^n(x-1))$ for $n \geq 2$ and $Y_1 = \operatorname{span}(x \mapsto x(x-1))$ instead.