Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 6

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Exercise 1. *Implement the numerical integration over a two-dimensional triangle τ using Gauss-Legendre quadrature.*
The Gauss-Legendre quadrature in the inverval $[0,1]$ has the weights w_{i,N_Q} and abscissae ξ_{i,N_Q} for $i = 1, 2, \cdots, N_Q$, which depend on the total number of quadrature points N_Q, and is exact for polynomials of degree $2N_Q - 1$ or less.
First of all, copy the weights and abscissae for $N_Q = 1, 2, \cdots, 10$ from the file glvalues.txt *on the website and store them in a suitable Matlab or Python function.*

(a) *Write a function for the one-dimensional Gauss-Legendre quadrature over $[0,1]$*

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{N_Q} w_{i,N_Q} f\left(\xi_{i,N_Q}\right).$$

(b) *Write a function for the two-dimensional Gauss-Legendre quadrature over $[0,1]^2$*

$$\int_0^1 \int_0^1 f(x,y)dxdy \approx \sum_{i=1}^{N_Q} \sum_{j=1}^{N_Q} w_{i,N_Q} w_{j,N_Q} f\left(\xi_{i,N_Q}, \eta_{j,N_Q}\right)$$

(c) *Approximate the following two integrals over any given triangle τ,*

$$I_\tau(g) := \int_\tau g(\mathbf{x})d\mathbf{x} \quad \text{and} \quad J_\tau(g) := \int_\tau \nabla g(\mathbf{x})d\mathbf{x}$$

by using a standard affine transformation onto the reference triangle $\hat{\tau}$ and the function

$$\begin{aligned} [0,1]^2 &\longrightarrow \hat{\tau}, \\ (\xi, \eta) &\longmapsto (\eta(1-\xi), \eta\xi). \end{aligned}$$