Numerical Methods for Elliptic and Parabolic Partial Differential Equations

Exercise Sheet 9

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Exercise 1 (Derive the posteriori estimate in 1D)**.**
Let $\Omega :=]0,1[$ *and* $\mathcal{T} := \{K_i : 1 \leqslant i \leqslant N\}$, *i.e.* $K_i := [x_{i-1}, x_i]$ *for some* $0 = x_0 < \cdots < x_N = 1$, $S := \left\{\varphi \in H_0^1(\Omega) \mid \forall K_i \in \mathcal{T} : \varphi|_{K_i} \in \mathbb{P}_1\right\}$. *Let* $f \in L^2(\Omega)$ *and suppose* u *satisfies the equation*

$$-u'' = f,$$

with homogeneous Dirichlet boundary condition and u_S *is the finite element approximation of* u *using* S. *Repeat the proof of (8.10) from manuscript to derive an estimate of the form* (1).[1] *How should the local quantities* $\eta_K(u_S)$ *be defined in this case?*

$$\|u - u_S\|_{H^1(\Omega)} \leqslant C \sqrt{\sum_{K \in \mathcal{T}} \eta_K^2(u_S)} \tag{1}$$

Hint. *You may use the existence of a Clément interpolation operator in 1D. This is an operator with the properties from Satz 8.1 in the lectur notes.*

Exercise 2. *Use Exercise 1 to implement an adaptive solution procedure for the following boundary value problem*

$$-u'' = f \quad \text{in } \Omega = (0,1), \quad u(0) = u(1) = 0, \quad f \in L^2(\Omega)$$

Hint: *The procedure should be implemented with iterative application of*

INITIALIZE: $\mathcal{T}_0 := [K_1, K_2]$ *where* $K_1 := \left[0, \frac{1}{2}\right]$ *and* $K_2 := \left[\frac{1}{2}, 1\right]$; *refinement parameter* $\alpha = 0.7$; *stopping criteria* $\varepsilon > 0$.

- *(a)* SOLVE: *assume* $\mathcal{T}_i$ *is generated. Solve the discrete problem for* $\mathcal{T}_i$, *hence obtain the approximation solution* u_i *on* $\mathcal{T}_i$;
- *(b)* ESTIMATE: *compute* η_K *for all* $K \in \mathcal{T}_i$ *and set* $\eta_{\max} = \max_{K \in \mathcal{T}_i} \eta_K(u_i)$; *if* $\eta_{\max} \leqslant \varepsilon$ *holds,* STOP; *otherwise:*
- *(c)* MARK: *mark all* K *for refinement which satisfy* $\eta_K \geqslant \alpha \eta_{\max}$;
- *(d)* REFINE: *bisect all marked intervals* $K \in \mathcal{T}_i$ *to obtain* $\mathcal{T}_{i+1}$, *increase* i *by 1 and go back to (a).*

Exercise 3. *Choose* $f = \frac{1}{x^{1/3}}$ *for the BVP in Exercise 1 and 2, verify that* $u = -\frac{9}{10} x \left(x^{2/3} - 1\right)$ *is the exact solution. Let your code run: depict the sequence of meshes, the error indicator and the exact error for each mesh* $\mathcal{T}_i$ *in the same picture.*

[1] This topic has been treated in the lecture in the week of November 19 and 20.