Numerical Methods for Elliptic and Parabolic Partial Differential Equations

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Exercise 1

Consider the heat equation

$$\dot{u} - \Delta u = 0,$$

with the initial condition $u(0) = v_0$, for some initial data $v_0 \in H_0^1(\Omega)$. Moreover, we enforce homogeneous Dirichlet boundary conditions (seek $u(\cdot, t) \in H_0^1(\Omega)$ for all $t > 0$. We discretize the problem in space, by the finite element method, which gives the following finite-dimensional evolution problem

$$\boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0)_j = v_0(x_j).$$

where $\boldsymbol{M}$ and $\boldsymbol{A}$ denote the mass and stiffness matrix, x_j the nodes of an underlying finite element mesh. The following constraints complete the problem formulation, but feel free to change any of them if you are interested in a different problem.

- Set $\Omega = [0, 1]^2$ and $v_0 = \sin(\pi x_1)\sin(\pi x_2)$.
- The exact solution is then given by $u(x, t) = e^{-2\pi^2 t}\sin(\pi x_1)\sin(\pi x_2)$.

In this setup, extend the implementation developed in the lecture to the heat equation, in the following ways:

- Use the implicit Euler method to approximate the solution of the heat equation.
- Use the explicit Euler method to approximate the solution to the heat equation. (Which only works for sufficiently small τ).

Use the finite element code from the lecture (in $2D$), or the last exercise (in $1D$) for the matrix assembly (to compute $\boldsymbol{M}$ and $\boldsymbol{A}$). If you choose to use the $1D$ code, you can easily derive a similar exact solution by setting $v_0 = \sin(\pi x)$ (on the interval $\Omega = [0, 1]$).