Exercise Sheet 9

Exercise 1 (Derive ~~a~~ the posteriori estimates in 1D.).

$\Omega :=]0,1[$

$\mathcal{T} := \{K_i : 1 \le i \le N\}$ ~~i.e.~~ a finite element mesh of Ω, i.e.

$K_i = [x_{i-1}, x_i]$ for some $0 = x_0 < \ldots < x_N = 1$,

$S := \{\varphi \in H^1_0(\Omega) : \forall K_i \in \mathcal{T} : \varphi|_{K_i} \in \mathbb{P}_1\}$

Let $f \in L^2(\Omega)$ & suppose u satisfies

$$u'' = f \quad \text{on } \Omega,$$

with $u(0) = u(1) = 0$, ~~&~~
u_S its finite element approximation of u using S.

Prove:

$$\|u - u_S\|_{H^1(\Omega)} \le C \sqrt{\sum_{K \in \mathcal{T}} \eta_K^2(u_S)},$$

how should the $\eta_K(u_S)$ be defined.

Hint: Repeat prove of (8.10), use Satz 8.1 in the lecture notes.

Proof

$a(\cdot,\cdot)$ is coercive $\Rightarrow \exists\, \alpha > 0 : \alpha \|u - u_S\|^2_{H^1(\Omega)}$
$\le a(u - u_S, u - u_S)$

$\underset{\text{by } \|u-u_S\|}{\overset{\text{dividing}}{\Rightarrow}} \quad \|u - u_S\|_{H^1(\Omega)} \le \frac{1}{\alpha} \sup_{\substack{v \in H^1_0(\Omega) \\ \|v\| \le 1}} a(u - u_S, v)$.

("this is called "stability").

. Goal: bound $a(u - u_S, v)$

We will need: Galerkin-Orthogonality

$$\forall v \in S : \quad a(u - u_S, v) = 0.$$

(this is true because $a(u,v) = l(v)$
$= a(u_S, v) \quad \forall s \in S$).

The map

$$v \in H^1_0(\Omega) \longmapsto a(u-u_S, v) = l(v) - a(u_S, v) \in \mathbb{R}$$

is a continuous linear functional, called "Residual"

Goal: we would like to bound the norm of this linear functional in order to obtain a bound for $\|u-u_S\|_{H^1(\Omega)}$.

To this end: let $v \in H^1_0(\Omega)$, $\|v\|_{H^1(\Omega)} = 1$.

$$\begin{aligned}
a(u-u_S, v) &= l(v) - a(u_S, v) \\
&= \int_\Omega f v - \int_\Omega v' u_S' \\
&= \sum_{K_i \in \mathcal{T}} \int_{K_i} f v - \int_{K_i} v' u_S' \\
&= \sum_{K_i \in \mathcal{T}} \int_{K_i} f v + \int_{K_i} u_S'' v - (u_S' v)\Big|_{x = x_{i-1}}^{x_i} \\
&= \sum_{K_i \in \mathcal{T}} \int_{K_i} f v + \int_{K_i} u_S'' v + \sum_{E \in \mathcal{E}} [u_S']_E \, v(E)
\end{aligned}$$

$\boxed{\text{let } \mathcal{E} := \{x_1, \dots, x_{N-1}\}. \quad [\varphi]_E := \lim_{h \to 0} \varphi(E+h) - \varphi(E-h)}$

~~later~~ Galerkin orthogonality $\Rightarrow$ we can ~~add~~ subtract any element $v_S \in S$ to v on the RHS.

$$\left.\begin{aligned}
\Rightarrow a(u-u_S, v) &= \sum_{K_i \in \mathcal{T}} \int_{K_i} f(v - v_S) + \int_{K_i} u_S''(v - v_S) \\
&\quad + \sum_{E \in \mathcal{E}} [u_S']_E \big(v(E) - v_S(E)\big) \\
&\overset{CS}{\leq} \sum_{K_i \in \mathcal{T}} \|f + u_S''\|_{L^2(K_i)} \|v - v_S\|_{L^2(K_i)} \\
&\quad + \sum_{E \in \mathcal{E}} |[u_S']_E| \, |v(E) - v_S(E)| .
\end{aligned}\right\} (*)$$

Next step: we ~~would~~ have to choose v_S in a way such that $\|v - v_S\|$ & $|v(E) - v_S(E)|$ are small.

~~We use: Quasi-Interpolation operator~~

$$\cancel{R_S : H^1_0(\Omega) \longrightarrow S^{1,0}_{g,0}}$$

~~for each $z \in \mathcal{E}$ define $\pi_z : L^2(K_z) \longrightarrow \mathbb{R}$, where~~

~~$K_z := K_{i-1} \cup K_i$ for $z = x_i$ with~~

$$\cancel{\pi_z \varphi := \frac{1}{|K_z|} \int_{K_z} \varphi}$$

~~Then~~

$$\cancel{R_S \varphi := \sum_{z \in \mathcal{E}} (\pi_z \varphi) b_z}$$ ~~where b_z is the hat function centered at z.~~

Hint: we can suppose there exists

$$R_S : H^1_0(\Omega) \longrightarrow S$$

s.t. $\exists\, C_1, C_2 > 0$ ~~(depending only~~ s.t. $\forall v \in H^1_0(\Omega)$

& $\forall K_i \in \mathcal{T}$ & $\forall E \in \mathcal{E}$: we suppose $K_{-1}, K_{N+1} = \emptyset$

$$\|v - R_S v\|_{L^2(K_i)} \leq C_1 h_i \|v\|_{H^1(K_{i-1} \cup K_i \cup K_{i+1})}$$

$$|v - R_S v|_E \leq C_2 \|v\|_{H^1(K_i \cup K_{i+1})} \quad \text{for } E = x_i$$

This is called Clément interpolation operator in 1D.

Continue (*): & set $v_S = R_S v$

$$a(u - u_S, v) \leq C_1 \sum_{K_i \in \mathcal{T}} h_i \|f + u_S''\|_{L^2(K_i)} \|v\|_{H^1(K_{i-1} \cup K_i \cup K_{i+1})}$$

$$+ C_2 \sum_{E \in \mathcal{E}} |[u_S']_E| \, \|v\|_{H^1(K_i \cup K_{i+1})}$$

CS

$$\leq C_1 \Big(\sum_{K_i \in \mathcal{T}} h_i^2 \|f + u_S''\|^2_{L^2(K_i)} \Big)^{\frac{1}{2}} \Big(\sum_{K_i \in \mathcal{T}} \|v\|^2_{H^1(K_{i-1} \cup K_i \cup K_{i+1})} \Big)^{\frac{1}{2}}$$

$$+ C_2 \Big(\sum_{E \in \mathcal{E}} |[u_S']_E|^2 \Big)^{1/2} \Big(\sum_{E \in \mathcal{E}} \|v\|^2_{H^1(K_i \cup K_{i+1})} \Big)^{\frac{1}{2}}$$

$(\cdot)^{1/2}$ is concave.

$$\leq C' \underbrace{\|v\|_{H^1(\Omega)}}_{=1} \Big(\sum_{K_i \in \mathcal{T}} h_i^2 \|f + u_S''\|^2_{L^2(K_i)} + \sum_{E \in \mathcal{E}} |[u_S']_E|^2 \Big)^{\frac{1}{2}}$$

$\Rightarrow \|u - u_S\|_{H^1(\Omega)} \leq C'\eta$

with $\eta := \left(\sum_{K_i \in \mathcal{T}} \eta_{K_i}^2 \right)^{1/2}$

& $\eta_{K_i}^2 := \left(h_i^2 \|f + u_S''\|_{L^2(K_i)}^2 + \frac{1}{2} \left(\left| [u_S']_{E=x_{i-1}} \right|^2 + \left| [u_S']_{E=x_i} \right|^2 \right) \right)^{1/2}$

$\blacksquare$