Mathematical Foundations for Finance

Exercise sheet 1

The first three exercises of this sheet contain material which is ***fundamental*** *for the course and* ***assumed to be known****. Please hand in your solutions by 12:00 on Wednesday, October 2 via the course homepage.*

Exercise 1.1 Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ be a finite set and $X : \Omega \to \mathbb{R}$ a mapping which takes the values $+5, 0$ and -5. You can think of X as a stock price change over one time period.

(a) What is the σ-field $\sigma(X)$ generated by X?

(b) Show that $|X|$ is measurable with respect to $\sigma\left(X^2\right)$.

(c) Let $Y : \Omega \to \mathbb{R}$ be another function. If $\sigma(Y) = 2^\Omega$, what can you say about Y?

Exercise 1.2 Consider a probability space $(\Omega, \mathcal{F}, P)$. A σ-algebra $\mathcal{F}_0 \subseteq \mathcal{F}$ is said to be P*-trivial* if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is P-trivial if and only if every $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$ is P-a.s. constant.

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then, the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E\left[X \mathbb{1}_A\right] = E[Z \mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of X given $\mathcal{G}$* and is denoted by $E\left[X \,|\, \mathcal{G}\right]$.
[*This is the formal definition of the conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.*]

(a) Show that if X is $\mathcal{G}$-measurable, then $E\left[X \,|\, \mathcal{G}\right] = X$ P-a.s.

(b) Show that $E\left[E\left[X \,|\, \mathcal{G}\right]\right] = E\left[X\right]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is P-trivial), then $E\left[X \,|\, \mathcal{G}\right] = E\left[X\right]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and some constants $a, b \in \mathbb{R}$. Show that $E\left[aX + bY \,|\, \mathcal{G}\right] = aE\left[X \,|\, \mathcal{G}\right] + bE\left[Y \,|\, \mathcal{G}\right]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e., there exists a collection $(A_i)_{i=1,\ldots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \ldots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \ldots, n$. Show that

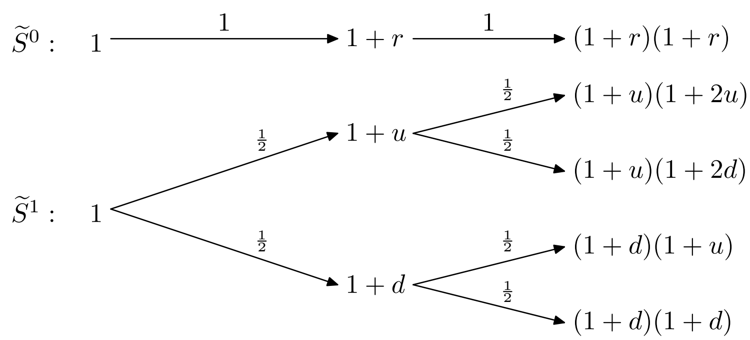
$$E\left[X \,|\, \mathcal{G}\right] = \sum_{i=1}^n E\left[X \,|\, A_i\right] \mathbb{1}_{A_i} \; P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated σ-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated σ-algebra, as for instance in the multinomial model.*]

Hint 1: Recall that $E[X \,|\, A_i] = E[X \mathbb{1}_{A_i}]/P[A_i]$ *and try to write* X *as a sum of random variables each of which only takes non-zero values on a single* A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ *has the form* $\cup_{j \in J} A_j$ *for some* $J \subseteq \{1, \ldots, n\}$.

Exercise 1.4 Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities:



Intuitively, this means that the volatility of $\tilde{S}^1$ increases after a stock price increase in the first period. Assume that u, $r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes $\tilde{S}^0$ and $\tilde{S}^1$ such that $\tilde{S}^1_k = \prod_{j=1}^{k} Y_j$ for $k = 0, 1, 2$.

(b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?

(c) For which values of u and d are Y_1 and Y_2 *independent*?

(d) For which values of u, r and d is the discounted stock process S^1 a P-martingale?