Mathematical Foundations for Finance

Exercise Sheet 10

Please hand in your solutions by 12:00 on Wednesday, December 4 via the course homepage.

Exercise 10.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider a Brownian motion W and a process $H \in L^2_{\text{loc}}(W)$. Let us denote the stochastic integral $I_t := \int_0^t H_s \mathrm{d}W_s$, for $t \geq 0$. Section 5.2 (*local martingale properties*) of the lecture notes shows that $(I_t)_{t\geq 0}$ is a local martingale. Prove that $(I_t)_{t\geq 0}$ is a martingale if any of the following conditions are satisfied:

(a) $(I_t)_{t\in[0,T]}$ is a martingale, for all $T \geq 0$;

(b) there exists $X \in L^1(P)$ such that $|I_t| \leq X$ for all $t \in [0, T]$, for all $T \geq 0$;
Hint: You may use dominated convergence theorem

(c) $E\left[\int_0^T H_s^2 \mathrm{d}s\right] < \infty$, for all $T \geq 0$;
Hint: You may prove that $H \in L^2(W^T)$ and use Proposition 5.1.4

Exercise 10.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider two independent Brownian motions W and B, and fix some constant $T > 0$.

(a) Consider the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t := \int_0^t s \mathrm{d}W_s + B_t.$$

Show that $X^T = (X_{t\wedge T})_{t\geq 0} \in \mathcal{M}_0^2$.
Hint: You may use the fact that if $M_1, M_2 \in \mathcal{M}_0^2$ then $M_1 + M_2 \in \mathcal{M}_0^2$.

(b) Prove that $[X]_t = t^3/3 + t$ P-a.s., for $t \geq 0$.

(c) Deduce that $E[(X_T)^2 \mid \mathcal{F}_t] = X_t^2 + \frac{T^3 - t^3}{3} + (T - t)$ P-a.s., for $t \in [0, T]$.

Exercise 10.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider a Brownian motion W. For any $t \geq 0$, using Itô's formula, write the following as stochastic integrals:

(a) W_t^2;

(b) $t^2 W_t$;

(c) $\sin(2t - W_t)$;

(d) $\exp(at + bW_t)$, where $a, b \in \mathbb{R}$ are constants.

Exercise 10.4 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ a filtration that satisfies the usual conditions. Let W be a Brownian motion on this space.

(a) Let $f \in C(\mathbb{R}; \mathbb{R})$. Show that the stochastic integral process $(\int_0^t f(W_s)\, \mathrm{d}W_s)_{t \geqslant 0}$ is a continuous local martingale.

(b) Let $f \in C^2(\mathbb{R}; \mathbb{R})$. Show that $(f(W_t))_{t \geqslant 0}$ is a continuous local martingale if and only if $\int_0^t f''(W_s)\, \mathrm{d}s = 0$ for all $t \geqslant 0$.

Hint: You may use the fact that a continuous local martingale null at zero is a process of finite variation if and only if it is identically 0.

(c) Using Itô's formula, establish which of the following processes are local martingales:

- $(\sin W_t - \cos W_t)_{t \geqslant 0}$;
- $(\exp(\frac{1}{2} a^2 t) \cos(aW_t - b))_{t \geqslant 0}$, where $a, b \in \mathbb{R}$ are constants;
- $(W_t^3 - 3tW_t)_{t \geqslant 0}$.