Mathematical Foundations for Finance

Exercise Sheet 10

Exercise 10.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider a Brownian motion W and a process $H \in L^2_{\text{loc}}(W)$. Let us denote the stochastic integral $I_t := \int_0^t H_s \mathrm{d}W_s$, for $t \geq 0$. Section 5.2 (*local martingale properties*) of the lecture notes shows that $(I_t)_{t\geq 0}$ is a local martingale. Prove that $(I_t)_{t\geq 0}$ is a martingale if any of the following conditions are satisfied:

(a) $(I_t)_{t\in[0,T]}$ is a martingale, for all $T \geq 0$;

(b) there exists $X \in L^1(P)$ such that $|I_t| \leq X$ for all $t \in [0,T]$, for all $T \geq 0$;
Hint: You may use dominated convergence theorem

(c) $E\left[\int_0^T H_s^2 \mathrm{d}s\right] < \infty$, for all $T \geq 0$;
Hint: You may prove that $H \in L^2(W^T)$ and use Proposition 5.1.4

Solution 10.1

(a) Adaptedness is clear. Then, notice that

$$[0,\infty) = \bigcup_{T\geq 0} [0,T].$$

Therefore, we can deduce that I_t is integrable for all $t \in [0,T]$, and all $T \geq 0$, if and only if I_t is integrable for every $t \in [0,\infty)$. Similarly, the martingale property $E[I_t \,|\, \mathcal{F}_s] = I_s$ P-a.s. holds for all $s \leq t$ with $s,t \in [0,T]$, and all $T \in [0,\infty)$, if and only if it holds for every $s \leq t$ such that $s,t \in [0,\infty)$.

(b) According to point (a), it suffices to prove that the condition in point (b) implies that $(I_t)_{t\in[0,T]}$ is a martingale, for a fixed $T > 0$. Since the process $(I_t)_{t\geq 0}$ is a local martingale, we can consider a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ for it. Then, $\tau_n := \sigma_n \wedge T$, for $n \in \mathbb{N}$, is a localizing sequence for $(I_t)_{t\in[0,T]}$. Moreover, by assumption, there exists a random variable $X \in L^1(P)$ such that

$$|I_t| = \left|\int_0^t H_s \mathrm{d}W_s\right| \leq X \ P\text{-a.s. for all } t \in [0,T].$$

Let us fix $0 \leq s \leq t \leq T$. We have that $\lim_{n\to\infty} \tau_n = T$ P-a.s., and thus

$$\lim_{n\to\infty} I_t^{\tau_n} = I_t \ P\text{-a.s. and } \lim_{n\to\infty} I_s^{\tau_n} = I_s \ P\text{-a.s.}$$

The dominated convergence theorem implies

$$E\left[I_t \,|\, \mathcal{F}_s\right] = E\left[\lim_{n\to\infty} I_t^{\tau_n} \,\middle|\, \mathcal{F}_s\right] = \lim_{n\to\infty} E\left[I_t^{\tau_n} \,|\, \mathcal{F}_s\right] = \lim_{n\to\infty} I_s^{\tau_n} = I_s \ P\text{-a.s.}$$

This concludes the proof that $(I_t)_{t\geq 0}$ is a martingale since adaptedness and integrability are clear.

(c) Let us fix $T \geq 0$. Since $(W_t^2 - t)_{t\geq 0}$ is a martingale, so is the process

$$(W_{t\wedge T}^2 - (t \wedge T))_{t\geq 0}$$

by Theorem 4.2.2 in the lecture notes. Moreover, the process $(t \wedge T)_{t\geq 0}$ is adapted to $\mathbb{F}$, increasing, null at 0 with $\Delta(t \wedge T) = 0 = (\Delta W_{t\wedge T})^2$. Theorem 5.1.1 allows us to conclude that

$$\left[W^T\right]_t = t \wedge T \ P\text{-a.s. for all } t \geq 0.$$

Hence,

$$E\left[\int_0^\infty H_s^2 \mathrm{d}\left[W^T\right]_s\right] = E\left[\int_0^\infty H_s^2 \mathrm{d}(s \wedge T)\right] = E\left[\int_0^T H_s^2 \mathrm{d}s\right] < \infty$$

by our assumption. Since it is clear that $W^T \in \mathcal{M}_0^2$, Proposition 5.1.4 directly implies that $\int H \mathrm{d}W^T \in \mathcal{M}_0^2$. Moreover,

$$\int_0^t H_s \mathrm{d}W_s^T = \left(\int_0^t H_s \mathrm{d}W_s\right)^T \ P\text{-a.s. for all } t \geq 0$$

because of Section 5.2 (*behaviour under stopping*) of the lecture notes. We deduce that

$$\left(\int H_s \mathrm{d}W_s\right)^T \in \mathcal{M}_0^2,$$

and in particular that $(\int_0^t H_s \mathrm{d}W_s)_{t\in[0,T]}$ is a square-integrable martingale. The proof follows by point (a).

Exercise 10.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider two independent Brownian motions W and B, and fix some constant $T > 0$.

(a) Consider the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t := \int_0^t s \mathrm{d}W_s + B_t.$$

Show that $X^T = (X_{t\wedge T})_{t\geq 0} \in \mathcal{M}_0^2$.
Hint: You may use the fact that if $M_1, M_2 \in \mathcal{M}_0^2$ then $M_1 + M_2 \in \mathcal{M}_0^2$.

(b) Prove that $[X]_t = t^3/3 + t$ P-a.s., for $t \geq 0$.

(c) Deduce that $E[(X_T)^2 \mid \mathcal{F}_t] = X_t^2 + \frac{T^3-t^3}{3} + (T-t)$ P-a.s., for $t \in [0,T]$.

Solution 10.2

(a) Section 5.2 (*behaviour under stopping*) of the lecture notes implies that

$$X_t^T = \left(\int_0^t s \mathrm{d}W_s\right)^T + B_t^T = \int_0^t s \mathrm{d}W_s^T + B_t^T \ P\text{-a.s. for all } t \geq 0.$$

It holds that $B^T \in \mathcal{M}_0^2$ since $E[(B_t^T)^2] = t \wedge T \leq T < \infty$ P-a.s., for $t \geq 0$. Moreover, the process $(H_s)_{s\geq 0}$ defined by $H_s := s$ is in $L^2(W^T)$ because it is clearly predictable since it is $\mathbb{F}$-adapted and continuous, and

$$E\left[\int_0^\infty s^2 \mathrm{d}\left[W^T\right]_s\right] = E\left[\int_0^\infty s^2 \mathrm{d}(s \wedge T)\right] = E\left[\int_0^T s^2 \mathrm{d}s\right] = \frac{T^3}{3} < \infty.$$

Proposition 5.1.4 implies that the stochastic integral $\left(\int_0^t s \mathrm{d}W_s^T\right) \in \mathcal{M}_0^2$. We can conclude that $X^T \in \mathcal{M}_0^2$.

(b) For any $t \geq 0$, Section 5.2 (*quadratic variation*) of the lecture notes implies

$$\begin{aligned} [X]_t &= \left[\int_0^\cdot s \mathrm{d}W_s + B\right]_t = \left[\int_0^\cdot s \mathrm{d}W_s\right]_t + 2\left[\int_0^\cdot s \mathrm{d}W_s, B\right]_t + [B]_t \\ &= \int_0^t s^2 \mathrm{d}[W]_s + 2\int_0^t s \mathrm{d}[W,B]_t + [B]_t = \frac{t^3}{3} + t \end{aligned}$$

since $[W]_t = [B]_t = t$, and $[W,B]_t = 0$ by the independence of W and B.

(c) Since $X^T \in \mathcal{M}_0^2$, the process $(X^T)^2 - [X]^T$ is a martingale. By point (b), we get that, for any $t \in [0,T]$,

$$E[(X_T)^2 \mid \mathcal{F}_t] = X_t^2 + E[[X]_T - [X]_t \mid \mathcal{F}_t] = X_t^2 + \frac{T^3 - t^3}{3} + (T-t) \ P\text{-a.s.}$$

Exercise 10.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Consider a Brownian motion W. For any $t \geq 0$, using Itô's formula, write the following as stochastic integrals:

(a) W_t^2;

(b) $t^2 W_t$;

(c) $\sin(2t - W_t)$;

(d) $\exp(at + bW_t)$, where $a, b \in \mathbb{R}$ are constants.

Solution 10.3

(a) Let $f(x) = x^2$. Then f is C^2, and $f'(x) = 2x$ and $f''(x) = 2$. Itô's formula then gives

$$W_t^2 = \int_0^t 2W_s \, \mathrm{d}W_s + \frac{1}{2}\int_0^t 2 \, \mathrm{d}s = 2\int_0^t W_s \, \mathrm{d}W_s + t.$$

(b) Let $f(t,x) = t^2 x$. Then f is C^2, and $\frac{\partial f}{\partial t}(t,x) = 2tx$, $\frac{\partial f}{\partial x}(t,x) = t^2$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = 0$. Itô's formula then gives

$$t^2 W_t = 2\int_0^t sW_s \, \mathrm{d}s + \int_0^t s^2 \, \mathrm{d}W_s.$$

(c) Let $f(t,x) = \sin(2t - x)$. Then f is C^2, and we have $\frac{\partial f}{\partial t}(t,x) = 2\cos(2t-x)$, $\frac{\partial f}{\partial x}(t,x) = -\cos(2t-x)$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = -\sin(2t-x)$. We then apply Itô's formula to get

$$\begin{aligned}
\sin(2t - W_t) &= \int_0^t 2\cos(2s - W_s) \, \mathrm{d}s - \int_0^t \cos(2s - W_s) \, \mathrm{d}W_s \\
&\quad - \frac{1}{2}\int_0^t \sin(2s - W_s) \, \mathrm{d}s \\
&= \int_0^t \Big(2\cos(2s - W_s) - \frac{1}{2}\sin(2s - W_s)\Big) \, \mathrm{d}s \\
&\quad - \int_0^t \cos(2s - W_s) \, \mathrm{d}W_s.
\end{aligned}$$

(d) Let $f(t,x) = \exp(at + bx)$. Then f is C^2, and $\frac{\partial f}{\partial t}(t,x) = a\exp(at+bx)$, $\frac{\partial f}{\partial x}(t,x) = b\exp(at+bx)$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = b^2\exp(at+bx)$. Itô's formula then gives

$$\begin{aligned}
\exp(at + bW_t) &= 1 + \int_0^t a\exp(as + bW_s) \, \mathrm{d}s + \int_0^t b\exp(as + bW_s) \, \mathrm{d}W_s \\
&\quad + \frac{1}{2}\int_0^t b^2 \exp(as + bW_s) \, \mathrm{d}s \\
&= 1 + \int_0^t \left(a + \frac{b^2}{2}\right)\exp(as + bW_s) \, \mathrm{d}s \\
&\quad + \int_0^t b\exp(as + bW_s) \, \mathrm{d}W_s,
\end{aligned}$$

as required.

Exercise 10.4 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that satisfies the usual conditions. Let W be a Brownian motion on this space.

(a) Let $f \in C(\mathbb{R};\mathbb{R})$. Show that the stochastic integral process $(\int_0^t f(W_s) \, \mathrm{d}W_s)_{t\geqslant 0}$ is a continuous local martingale.

(b) Let $f \in C^2(\mathbb{R};\mathbb{R})$. Show that $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $\int_0^t f''(W_s)\ \mathrm{d}s = 0$ for all $t \geqslant 0$.

Hint: You may use the fact that a continuous local martingale null at zero is a process of finite variation if and only if it is identically 0.

(c) Using Itô's formula, establish which of the following processes are local martingales:

- $(\sin W_t - \cos W_t)_{t\geqslant 0}$;
- $(\exp(\frac{1}{2}a^2 t)\cos(aW_t - b))_{t\geqslant 0}$, where $a, b \in \mathbb{R}$ are constants;
- $(W_t^3 - 3tW_t)_{t\geqslant 0}$.

Solution 10.4

(a) First note that $(f(W_s))_{s\geqslant 0}$ is adapted (since W is adapted and f is continuous) with continuous paths (since W has continuous paths and f is continuous). In particular, $(f(W_s))_{s\geqslant 0}$ is predictable and locally bounded, and thus belongs to $L^2_{\mathrm{loc}}(W)$. Since W is a (local) martingale null at zero, the stochastic integral process $(\int_0^t f(W_s)\ \mathrm{d}W_s)_{t\geqslant 0}$ is thus a well-defined continuous local martingale.

(b) By Itô's formula,

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\ \mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s)\ \mathrm{d}s\ P\text{-a.s., for } t \geq 0.$$

By part (a), we know that $(\int_0^t f'(W_s)\ \mathrm{d}W_s)_{t\geqslant 0}$ is a continuous local martingale, and thus $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is also a continuous local martingale. But $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is a process of finite variation (indeed, for each $t \geqslant 0$, we have the equality $\int_0^t f''(W_s)\ \mathrm{d}s = \int_0^t f''(W_s)^+\ \mathrm{d}s - \int_0^t f''(W_s)^-\ \mathrm{d}s$, so that $(\int_0^t f''(W_s)\ \mathrm{d}s)_{t\geqslant 0}$ is the difference of two increasing processes), null at zero, and is thus a continuous local martingale if and only if it is identically zero. That is, $(f(W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if $\int_0^t f''(W_s)\ \mathrm{d}s = 0$ for all $t \geqslant 0$, as required.

(c) By the same reasoning as in point (b), we can show using Itô's formula that for $f \in C^2([0,\infty)\times\mathbb{R};\mathbb{R})$, the process $(f(t, W_t))_{t\geqslant 0}$ is a continuous local martingale if and only if

$$\int_0^t \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\ \mathrm{d}s = 0\ P\text{-a.s., for any } t \geqslant 0.$$

- Let $f(x) = \sin x - \cos x$. Then f is C^2, and $f(W_t) = \sin W_t - \cos W_t$. Since $f''(x) = -\sin x + \cos x$, then $f'' \not\equiv 0$, and thus $(\sin W_t - \cos W_t)_{t\geqslant 0}$ is not a local martingale.

- Let $f(t,x) = \exp(\frac{1}{2}a^2 t)\cos(ax-b)$. Then f is C^2, and also we have that $\frac{\partial f}{\partial t}(t,x) = \frac{1}{2}a^2 \exp(\frac{1}{2}a^2 t)\cos(ax-b)$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = -a^2\exp(\frac{1}{2}a^2 t)\cos(ax-b)$. Thus,

$$\begin{aligned}&\int_0^t \frac{\partial f}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,W_s)\ \mathrm{d}s\\&= \int_0^t \frac{1}{2}a^2\exp\left(\frac{1}{2}a^2 t\right)\cos(aW_t-b) - \frac{1}{2}a^2\exp\left(\frac{1}{2}a^2 t\right)\cos(aW_t-b)\ \mathrm{d}s\\&= 0\ P\text{-a.s., for any } t \geq 0.\end{aligned}$$

  Since $\exp(\frac{1}{2}a^2 t)\cos(aW_t-b) = f(t,W_t)$, it now follows that $(\exp(\frac{1}{2}a^2 t)\cos(aW_t-b))_{t\geqslant 0}$ is a continuous local martingale.

- Let $f(t,x) = x^3 - 3tx$. Then f is C^2, and $\frac{\partial f}{\partial t}(t,x) = -3x$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = 6x$. For any $t \geq 0$, we then have that

$$\int_0^t \frac{\partial f}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,W_s)\ \mathrm{d}s = \int_0^t 3W_s - 3W_s\ \mathrm{d}s = 0\ P\text{-a.s.}$$

  Since $W_t^3 - 3tW_t = f(t,W_t)$, it follows that $(W_t^3 - 3tW_t)_{t\geqslant 0}$ is a continuous local martingale.