Coordinator
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Mathematical Foundations for Finance

Exercise Sheet 12

Please hand in your solutions by 12:00 on Wednesday, December 18 via the course homepage.

Exercise 12.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Assume that $\mathcal{F}_0$ is P-trivial and consider a Brownian motion W on this space.

(a) Prove that any continuous, adapted process H is predictable and locally bounded.

Hint: Recall that a process X is locally bounded if there is a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity such that X^{τ_n} is uniformly bounded P-a.s.

(b) Prove that any predictable, locally bounded process H is an element of $L^2_{\mathrm{loc}}(W)$.

Exercise 12.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$\mathrm{d}\widetilde{S}^0_t = \widetilde{S}^0_t r\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t\Big(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\Big), \tag{1}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic.

(a) Prove using Itô's formula and (1) that the discounted stock price process $S^1 = \widetilde{S}^1/\widetilde{S}^0$ solves

$$\mathrm{d}S^1_t = S^1_t\Big((\mu - r)\mathrm{d}t + \sigma\mathrm{d}W_t\Big). \tag{2}$$

(b) Prove using Itô's formula that

$$S^1 = \left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right)\right)_{t\in[0,T]},$$

i.e. show that the process $\left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right)\right)_{t\in[0,T]}$ solves (2).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := \Big(W_t + \lambda t\Big)_{t\in[0,T]}$ is a Brownian motion under the measure Q_λ given by $\frac{\mathrm{d}Q_\lambda}{\mathrm{d}P} := Z^\lambda_T$.

(d) Prove that for the right choice of λ, the discounted stock price process S^1 is a Q_λ-martingale.

Hint: Rewrite $\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t$ *as a function of* $W_t^\lambda, t, \sigma, \mu,$ *and* r.

Exercise 12.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$\frac{\mathrm{d}\widetilde{S}^0_t}{\widetilde{S}^0_t} = r\,\mathrm{d}t \quad \text{and} \quad \frac{\mathrm{d}\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$.

Hint: If you did not find λ^* *in the Exercise 12.2(d), you can use that* $\lambda^* := \frac{\mu - r}{\sigma}$.

(a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \widehat{=} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u \,\mathrm{d}S^1_u = \frac{(\widetilde{S}^1_T)^2}{\widetilde{S}^0_T}.$$

Hint: Look for a representation result under Q^**, not under* P.

(b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi \widehat{=} (\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u \,\mathrm{d}S^1_u = \frac{1}{\widetilde{S}^0_T \widetilde{S}^1_T}.$$

Exercise 12.4 A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\ge 0}$ which is adapted to $\mathbb{F}$, has $N_0 = 0$ P-a.s. and satisfies the following two properties:

(PP1) For $0 \le s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed* with parameter $\lambda(t-s)$, i.e.

$$P[N_t - N_s = k] = e^{-\lambda(t-s)} \frac{\big(\lambda(t-s)\big)^k}{k!}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modelling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.*
Hint: If $X \sim Poi(\lambda)$*, then* $E[X] = \lambda$.

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N} \rangle$.
Hint: If $X \sim Poi(\lambda)$*, then* $\mathrm{Var}[X] = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda \sigma t}$, $t \geq 0$, where $\sigma > -1$. S is also called a *geometric Poisson process.*