Mathematical Foundations for Finance

Exercise Sheet 12

Exercise 12.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Assume that $\mathcal{F}_0$ is P-trivial and consider a Brownian motion W on this space.

(a) Prove that any continuous, adapted process H is predictable and locally bounded.

Hint: Recall that a process X is locally bounded if there is a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity such that X^{τ_n} is uniformly bounded P-a.s.

(b) Prove that any predictable, locally bounded process H is an element of $L^2_{\mathrm{loc}}(W)$.

Solution 12.1

(a) Recall that a process H is predictable if it is $\mathcal{P}$-measurable when viewed as a mapping $H : \overline{\Omega} \to \mathbb{R}$, for $\overline{\Omega} := \Omega \times (0, \infty)$ and $\mathcal{P}$ being the σ-field on $\overline{\Omega}$ generated by all left-continuous adapted processes. Since H is adapted and continuous (therefore also left-continuous), it is obviously predictable.

Define now $(\tau_n)_{n\in\mathbb{N}}$ as

$$\tau_n := \inf\{t \geq 0 \mid |H_t| > n\}$$

for all $n \in \mathbb{N}$. Observe that τ_n is a stopping time for all $n \in \mathbb{N}$ by the continuity of H and the right-continuity of the filtration. The sequence $(\tau_n)_{n\in\mathbb{N}}$ is then clearly increasing P-a.s. since the Brownian Motion has P-a.s. continuous trajectories.

Fix now $\omega \in \Omega$ such that the map $t \mapsto H_t(\omega)$ is continuous. Since continuous functions are bounded on compact intervals, we have that for all $T \geq 0$, there exists an $N := N(\omega, T) \in \mathbb{N}$ such that $|H_t(\omega)| < N$ for all $t \in [0, T]$, and thus $\tau_n(\omega) \geq T$ for all $n \geq N$. As a result $\lim_{n\to\infty} \tau_n(\omega) = \infty$ and hence $\lim_{n\to\infty} \tau_n = \infty$ P-a.s. We can thus conclude that $(\tau_n)_{n\in\mathbb{N}}$ defines a localizing sequence.

Finally, by definition of τ_n, we have that for all $\omega \in \Omega$,

$$|H_t(\omega)| \leq n \qquad \forall\ t < \tau_n(\omega).$$

Moreover, there are two possible cases. Either $\tau_n(\omega) = 0$ and hence $|H_{\tau_n(\omega)}(\omega)| = |H_0(\omega)|$, or $\tau_n(\omega) > 0$ and hence $[0, \tau_n(\omega)) \neq \emptyset$ and by continuity of H we can compute

$$|H_{\tau_n(\omega)}(\omega)| = \lim_{\substack{t \to \tau_n(\omega) \\ t < \tau_n(\omega)}} |H_t(\omega)| \leq n$$

for P-a.a. $\omega \in \Omega$. Since $\mathcal{F}_0$ is P-trivial, $H_0 = h_0 \in \mathbb{R}$ P-a.s. and we can conclude that $|H_t(\omega)| \leq n \vee |h_0|$ for all $t \leq \tau_n(\omega)$ and P-a.a. $\omega \in \Omega$ and thus

$$|H_t^{\tau_n}| \leq n \vee |h_0| \qquad P\text{-a.s., for all } t \geq 0. \tag{1}$$

(b) Since W is a *continuous* (local) martingale, $H \in L^2_{\mathrm{loc}}(W)$ if and only if it is predictable and

$$\int_0^t H_s^2 \mathrm{d}\langle W \rangle_s = \int_0^t H_s^2 \mathrm{d}s < \infty \qquad P\text{-a.s.}$$

for each $t \geq 0$. The first property is true by assumption. For the second one, let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times increasing P-a.s. to infinity such that H^{τ_n} is uniformly bounded P-a.s. (i.e. $|H_t^{\tau_n}| \leq c_n$ for some $c_n \geq 0$, for all $t \geq 0$).

Let Ω_0 be the set of all $\omega \in \Omega$ such that $\lim_{n \to \infty} \tau_n(\omega) = \infty$ and $|H_t^{\tau_n}(\omega)| \leq c_n$ for all $t \geq 0$ and $n \in \mathbb{N}$. Since countable intersections of sets of probability one are of probability 1, $P[\Omega_0] = 1$. Fix then $\omega \in \Omega_0$ and $t > 0$. Observe that since $\lim_{n \to \infty} \tau_n(\omega) = \infty$, there exists an $N := N(\omega, t) \in \mathbb{N}$ such that $\tau_N(\omega) > t$. As a result

$$\int_0^t H_s^2(\omega) \mathrm{d}s = \int_0^t \Big(H_s^{\tau_N}(\omega)\Big)^2 \mathrm{d}s \leq \int_0^t c_N^2 \mathrm{d}s = c_N^2 t < \infty$$

and hence $\int_0^t H_s^2(\omega) \mathrm{d}s < \infty$ for all $\omega \in \Omega_0$ and $t > 0$.

Exercise 12.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t \in [0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t \in [0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t \in [0,T]}$ are given by

$$\mathrm{d}\widetilde{S}^0_t = \widetilde{S}^0_t r \, \mathrm{d}t \quad \text{and} \quad \mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t \Big(\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t\Big), \tag{2}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic.

(a) Prove using Itô's formula and (2) that the discounted stock price process $S^1 = \widetilde{S}^1 / \widetilde{S}^0$ solves

$$\mathrm{d}S^1_t = S^1_t \Big((\mu - r)\mathrm{d}t + \sigma \mathrm{d}W_t\Big). \tag{3}$$

(b) Prove using Itô's formula that

$$S^1 = \left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right)\right)_{t \in [0,T]},$$

i.e. show that the process $\left(S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right)\right)_{t\in[0,T]}$ solves (3).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := \left(W_t + \lambda t\right)_{t\in[0,T]}$ is a Brownian motion under the measure Q_λ given by $\frac{\mathrm{d}Q_\lambda}{\mathrm{d}P} := Z_T^\lambda$.

(d) Prove that for the right choice of λ, the discounted stock price process S^1 is a Q_λ-martingale.

Hint: Rewrite $\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t$ *as a function of* $W_t^\lambda, t, \sigma, \mu,$ *and* r.

Solution 12.2

(a) Using that $\widetilde{S}^0 > 0$ P-a.s., we can apply Itô's formula to the C^2-function $f : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ given by $f(x,y) := x/y$. Computing the different derivatives, we obtain

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(x,y) = 0,$$

and moreover, since $\widetilde{S}^0$ is of finite variation,

$$\langle \widetilde{S}^1, \widetilde{S}^0\rangle = 0 \qquad \text{and} \qquad \langle \widetilde{S}^0, \widetilde{S}^0\rangle = 0.$$

By Itô's formula, we obtain

$$\begin{aligned} S_t^1 = \frac{\widetilde{S}_t^1}{\widetilde{S}_t^0} &= S_0^1 + \int_0^t \frac{1}{\widetilde{S}_s^0}\mathrm{d}\widetilde{S}_s^1 + \int_0^t \frac{-\widetilde{S}_s^1}{(\widetilde{S}_s^0)^2}\mathrm{d}\widetilde{S}_s^0 + 0 \\ &= S_0^1 + \int_0^t \frac{1}{\widetilde{S}_s^0}\left(\widetilde{S}_s^1\left(\mu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s\right)\right) + \int_0^t \frac{-\widetilde{S}_s^1}{(\widetilde{S}_s^0)^2}\left(\widetilde{S}_s^0 r\,\mathrm{d}s\right) \\ &= S_0^1 + \int_0^t S_s^1\left((\mu - r)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s\right), \end{aligned}$$

or written equivalently in the differential notation,

$$dS_t^1 = S_t^1\left((\mu - r)dt + \sigma dW_t\right).$$

(b) Consider the C^2-function $f : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ given by $f(w,t) := S_0^1 \exp\left(\sigma w + (\mu - r - \sigma^2/2)t\right)$. Computing the different derivatives, we obtain

$$\frac{\partial f}{\partial w}(w,t) = \sigma f(w,t), \quad \frac{\partial f}{\partial t}(w,t) = (\mu - r - \sigma^2/2)f(w,t), \quad \text{and} \quad \frac{\partial^2 f}{\partial w^2}(w,t) = \sigma^2 f(w,t),$$

and moreover, since $(t)_{t\in[0,T]}$ is of finite variation,

$$\langle W, W\rangle_t = t, \qquad \langle t, W\rangle = \langle W, t\rangle = 0, \qquad \text{and} \qquad \langle t, t\rangle = 0.$$

By Itô's formula, we can then compute

$$f(W_t,t) = S_0^1 + \int_0^t \sigma f(W_s,s)\mathrm{d}W_s + \int_0^t \Big(\mu - r - \frac{\sigma^2}{2}\Big) f(W_s,s)\mathrm{d}s + \frac{1}{2}\int_0^t \sigma^2 f(W_s,s)\mathrm{d}s$$
$$= S_0^1 + \int_0^t f(W_s,s)\Big(\sigma\mathrm{d}W_s + \big(\mu - r\big)\mathrm{d}s\Big),$$

and thus conclude that $f(W_t,t) = S_0^1 \exp\Big(\sigma W_t + (\mu - r - \sigma^2/2)t\Big)$ solves the SDE.

(c) Since $\lambda t = \int_0^t \lambda \mathrm{d}s$, we can write $W_t^\lambda = W_t + \int_0^t \lambda \mathrm{d}s$. Using that $L_t = -\lambda W_t = \int_0^t -\lambda \mathrm{d}W_s$, we can then directly conclude by Girsanov's theorem (Theorem 6.2.3 in the lecture notes), that W^λ is a Q_λ-Brownian motion.

(d) Note that we can write

$$\sigma W_t + \Big(\mu - r - \frac{1}{2}\sigma^2\Big)t = \sigma(W_t^\lambda - \lambda t) + \Big(\mu - r - \frac{1}{2}\sigma^2\Big)t = \sigma W_t^\lambda + \Big(\mu - r - \frac{1}{2}\sigma^2 - \sigma\lambda\Big)t.$$

Since by point (c) W^λ is a Brownian motion under Q_λ, we can deduce from Proposition 4.2.2 in the lecture notes that the process

$$\left(S_0^1 \exp\left(\sigma W_t + \Big(\mu - r - \frac{1}{2}\sigma^2\Big)t\right)\right)_{t\in[0,T]} \tag{4}$$

is a Q_λ-martingale if and only if $\mu - r - \frac{1}{2}\sigma^2 - \sigma\lambda = -\frac{1}{2}\sigma^2$. Solving this equation, we can conclude by (b), since S^1 coincides with the process in (4), that S^1 is a Q^λ martingale if and only if $\lambda = \frac{\mu - r}{\sigma}$.

Exercise 12.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}_t^0)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t\in[0,T]}$ are given by

$$\frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} = r\,\mathrm{d}t \quad \text{and} \quad \frac{\mathrm{d}\widetilde{S}_t^1}{\widetilde{S}_t^1} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}_0^0 = 1$ and $\widetilde{S}_0^1 > 0$ are deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$.

Hint: If you did not find λ^* in the Exercise 12.2(d), you can use that $\lambda^* := \frac{\mu - r}{\sigma}$.

(a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \widehat{=} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u\,\mathrm{d}S_u^1 = \frac{(\widetilde{S}_T^1)^2}{\widetilde{S}_T^0}.$$

Hint: Look for a representation result under Q^*, not under P.

(b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi\widehat{=}(\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u \, \mathrm{d}S_u^1 = \frac{1}{\widetilde{S}_T^0 \widetilde{S}_T^1}.$$

Solution 12.3 Set $\lambda^* := \frac{\mu - r}{\sigma}$. We know from the previous exercise that $W_t^* := W_t + \lambda^* t$, $t \in [0, T]$ is a Brownian motion under the equivalent martingale measure Q^*. Moreover, the discounted stock price process $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$ is a Q^*-martingale and is explicitly given by

$$S_t = S_0^1 e^{\sigma W_t^* - \frac{1}{2}\sigma^2 t}.$$

The *discounted* arbitrage-free value at time t of any *discounted* payoff $H \in L_+^1(\mathcal{F}_T, Q^*)$ is given by

$$V_t^* = E_{Q^*}\left[H \,|\, \mathcal{F}_t\right].$$

(a) We use the discussion from page 139 of the lecture notes to conclude that $V_t^* = E_{Q^*}\left[H \,|\, \mathcal{F}_t\right]$ may be represented as a stochastic integral of the form

$$V_t^* = E_{Q^*}\left[H\right] + \int_0^t \varphi_s \, \mathrm{d}S_s^1, \quad 0 \le t \le T.$$

In this case, we compute

$$\begin{aligned} V_t^* &= e^{-rT} E_{Q^*}\left[\left(\widetilde{S}_T^1\right)^2 \middle| \mathcal{F}_t\right] = e^{-rT} e^{2Tr} E_{Q^*}\left[\left(S_T^1\right)^2 \middle| \mathcal{F}_t\right] \\ &= e^{rT}\left(S_t^1\right)^2 E_{Q^*}\left[e^{2\sigma(W_T^* - W_t^*) - \sigma^2(T-t)} \middle| \mathcal{F}_t\right] \\ &= e^{(r+\sigma^2)T - \sigma^2 t}\left(S_t^1\right)^2 =: v\left(t, S_t^1\right). \end{aligned} \tag{5}$$

We apply Itô's formula to v and obtain

$$v(t, S_t^1) = v(0, S_0^1) + \int_0^t \frac{\partial}{\partial x} v(t, S_t^1) \, \mathrm{d}S_t^1 + \text{ continuous FV process.}$$

Since the left-hand side and the stochastic integral on the right-hand side are local $(Q^*, \mathbb{F})$-martingales, the "continuous FV process" is a local $(Q^*, \mathbb{F})$-martingale as well and since it apparently is null at 0, it must be identically equal to 0. Thus, it must vanish identically. We thus immediately obtain that

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1) = 2e^{(r+\sigma^2)T - \sigma^2 t} S_t^1 = 2e^{(r+\sigma^2)T + (r-\sigma^2)t} \widetilde{S}_t^1.$$

For $v(0, S_0^1)$, we have that $v(0, S_0^1) = e^{(r+\sigma^2)T}(S_0^1)^2$.

(b) We proceed analogously as in part (a). Here, we obtain

$$\begin{aligned}\overline{V}_t^* &= e^{-rT} E_{Q^*}\left[\frac{1}{\widetilde{S}_T^1}\,\middle|\,\mathcal{F}_t\right] = e^{-2rT}\frac{1}{S_t^1} E_{Q^*}\left[\frac{S_t^1}{S_T^1}\,\middle|\,\mathcal{F}_t\right] \\ &= e^{-2rT}\frac{1}{S_t^1} E_{Q^*}\left[e^{-\sigma(W_T^*-W_t^*)+\frac{1}{2}\sigma^2(T-t)}\,\middle|\,\mathcal{F}_t\right] \\ &= e^{(\sigma^2-2r)T-\sigma^2 t}\frac{1}{S_t^1} \\ &=: \overline{v}(t, S_t^1)\,.\end{aligned}$$

We conclude again in the same way as in (a) that

$$\begin{aligned}\overline{\vartheta}_t &= \frac{\partial}{\partial x}\overline{v}(t, S_t^1) = -e^{(\sigma^2-2r)T-\sigma^2 t}\frac{1}{(S_t^1)^2} = -e^{(\sigma^2-2r)(T-t)}\frac{1}{(\widetilde{S}_t^1)^2} \\ \overline{V}_0^* &= v(0, S_0^1) = e^{(\sigma^2-2r)T}\frac{1}{S_0^1} = e^{(\sigma^2-2r)T}\frac{1}{\widetilde{S}_0^1}.\end{aligned}$$

Exercise 12.4 A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, has $N_0 = 0$ P-a.s. and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed* with parameter $\lambda(t-s)$, i.e.

$$P[N_t - N_s = k] = e^{-\lambda(t-s)}\frac{\big(\lambda(t-s)\big)^k}{k!}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modelling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.* *Hint: If* $X \sim$ *Poi*(λ), *then* $E[X] = \lambda$.

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N} \rangle$. *Hint: If* $X \sim$ *Poi*(λ), *then* $\operatorname{Var}[X] = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda\sigma t}$, $t \geq 0$, where $\sigma > -1$. S is also called a *geometric Poisson process.*

Solution 12.4 In all three cases, adaptedness is obvious and integrability is also clear, since each $N_t = N_t - N_0 \sim \text{Poi}(\lambda t)$ has a Poisson distribution, which has finite exponential moments and hence also finite moments of all orders. What remains to be shown in all cases is the *martingale property*. Let $0 \leq s < t$.

(a) Using that $N_t - N_s \sim \text{Poi}(\lambda(t-s))$ is independent of $\mathcal{F}_s$, we get

$$E\left[N_t - N_s \,|\, \mathcal{F}_s\right] = E\left[N_t - N_s\right] = \lambda(t-s) = \lambda t - \lambda s \quad P\text{-a.s.}$$

Since N_s is $\mathcal{F}_s$-measurable, we can rearrange the above equation to obtain

$$E\left[N_t - \lambda t \,|\, \mathcal{F}_s\right] = N_s - \lambda s \quad P\text{-a.s.},$$

which is what we wanted to show.

(b) For any square-integrable martingale M, we have

$$E\left[M_t^2 - M_s^2 \,\middle|\, \mathcal{F}_s\right] = E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right] \quad \text{for } s \leq t.$$

Indeed,

$$\begin{aligned} E\left[M_t^2 - M_s^2 \,\middle|\, \mathcal{F}_s\right] &= E\left[M_t^2 - 2M_sM_t + M_s^2 + 2M_sM_t - 2M_s^2 \,\middle|\, \mathcal{F}_s\right] \\ &= E\left[(M_t - M_s)^2 + 2M_sM_t - 2M_s^2 \,\middle|\, \mathcal{F}_s\right] \\ &= E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right] + 2M_sE\left[M_t - M_s \,|\, \mathcal{F}_s\right] \\ &= E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right]. \end{aligned}$$

Using this for $M = \widetilde{N}$ gives

$$\begin{aligned} E\left[\widetilde{N}_t^2 - \widetilde{N}_s^2 \,\middle|\, \mathcal{F}_s\right] &= E\left[\left(\widetilde{N}_t - \widetilde{N}_s\right)^2 \,\middle|\, \mathcal{F}_s\right] = E\left[(N_t - N_s - \lambda(t-s))^2 \,\middle|\, \mathcal{F}_s\right] \\ &= E\left[(N_t - N_s - E\left[N_t - N_s\right])^2\right] \\ &= \text{Var}(N_t - N_s) = \lambda(t-s). \end{aligned}$$

Since $\widetilde{N}_s^2$ is $\mathcal{F}_s$-measurable, we can rearrange this to obtain that

$$E\left[\widetilde{N}_t^2 - \lambda t \,\middle|\, \mathcal{F}_s\right] = \widetilde{N}_s^2 - \lambda s,$$

which gives the martingale property for the process $(\widetilde{N}_t^2 - \lambda t)_{t \geq 0}$.

Using the previous result, we can also easily compute that

$$\begin{aligned} E\left[\widetilde{N}_t^2 - N_t - (\widetilde{N}_s^2 - N_s) \,\middle|\, \mathcal{F}_s\right] &= E\left[\widetilde{N}_t^2 - \widetilde{N}_s^2 - (N_t - N_s) \,\middle|\, \mathcal{F}_s\right] \\ &= E\left[\widetilde{N}_t^2 - \widetilde{N}_s^2 \,\middle|\, \mathcal{F}_s\right] - E\left[N_t - N_s \,|\, \mathcal{F}_s\right] \\ &= \lambda(t-s) - \lambda(t-s) = 0, \end{aligned}$$

giving the martingale property for the process $(\widetilde{N}_t^2 - N_t)_{t\geq 0}$. In addition, N is null at zero, adapted to $\mathbb{F}$, increasing and we have that

$$\Delta N = (\Delta N)^2 = (\Delta \widetilde{N})^2$$

because all jumps of N are of size 1. By Theorem 5.1.1, we therefore have that $[\widetilde{N}] = N$. Additionally, the process $(\lambda t)_{t\geq 0}$ is null at 0, predictable, increasing, and we have that

$$[\widetilde{N}]_t - \lambda t = N_t - \lambda t$$

is a (local) martingale, which means that $\langle \widetilde{N} \rangle_t = \lambda t$.

(c) If $X \sim \mathrm{Poi}(\mu)$ and $a > 0$, we have that

$$E\left[e^{aX}\right] = \sum_{k=0}^{\infty} e^{ak} \frac{\mu^k}{k!} e^{-\mu} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(e^a \mu)^k}{k!} = e^{-\mu} e^{e^a \mu} = e^{\mu(e^a - 1)}.$$

Using this result and the fact that $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$ is independent of $\mathcal{F}_s$, we get

$$\begin{aligned} E\left[\frac{S_t}{S_s} \,\middle|\, \mathcal{F}_s\right] &= E\left[e^{(N_t - N_s)\log(1+\sigma) - \lambda\sigma(t-s)} \,\middle|\, \mathcal{F}_s\right] \\ &= e^{-\lambda\sigma(t-s)} E\left[e^{(N_t - N_s)\log(1+\sigma)}\right] \\ &= e^{-\lambda\sigma(t-s)} e^{\lambda(t-s)(1+\sigma-1)} = 1 \quad P\text{-a.s.} \end{aligned}$$