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# Mathematical Foundations for Finance

## Exercise sheet 1

**Exercise 1.1** Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ be a finite set and $X : \Omega \to \mathbb{R}$ a mapping which takes the values $+5, 0$ and $-5$. You can think of $X$ as a stock price change over one time period.

(a) What is the $\sigma$-field $\sigma(X)$ generated by $X$?

(b) Show that $|X|$ is measurable with respect to $\sigma\left(X^2\right)$.

(c) Let $Y : \Omega \to \mathbb{R}$ be another function. If $\sigma(Y) = 2^\Omega$, what can you say about $Y$?

**Solution 1.1**

(a) We have

$$\begin{aligned}\sigma(X) &= \sigma\left(\{X=5\}, \{X=0\}, \{X=-5\}\right) \\ &= \big\{\emptyset, \Omega, \{X=5\}, \{X=0\}, \{X=-5\}, \\ &\qquad \{X=5\} \cup \{X=0\}, \{X=-5\} \cup \{X=0\}, \{X=-5\} \cup \{X=5\}\big\}\,.\end{aligned}$$

The second equality follows from the fact that the last system of sets is a $\sigma$-field and contains a generator of $\sigma(X)$. Thus by definition, they have to be equal.

(b) Since $|X| = \sqrt{X^2}$, $|X|$ is a continuous function of $X^2$, hence $\sigma\left(X^2\right)$-measurable. One could also argue by explicitly writing out the $\sigma$-field $\sigma\left(X^2\right)$ as in a). One gets

$$\begin{aligned}\sigma\left(X^2\right) &= \big\{\emptyset, \Omega, \{X^2=25\}, \{X^2=0\}, \{X^2=0\} \cup \{X^2=25\}\big\} \\ &= \{\emptyset, \Omega, \{|X|=5\}, \{|X|=0\}\}\ .\end{aligned}$$

It follows immediately that $|X|$ is $\sigma\left(X^2\right)$-measurable.

(c) Because $\Omega$ is finite, $Y$ can take at most $N$ different values. Therefore $\sigma(Y)$ is finite and generated by the sets of the form $\{Y = y_i\}$ for a finite collection of numbers $y_1, y_2, \ldots, y_n \in \mathbb{R}$, $n \le N$. The $\sigma$-field generated by these sets has exactly $2^n$ elements. The power set $2^\Omega$ of $\Omega$ has $2^N$ elements. Hence $Y$ must take a different value on each one of the $\omega_1, \omega_2, \ldots, \omega_n$, and so $N = n$. In summary, then, we can say that $Y$ takes a different value on each $\omega_i$, $i = 1, \ldots, N$.

**Exercise 1.2** Consider a probability space $(\Omega, \mathcal{F}, P)$. A $\sigma$-algebra $\mathcal{F}_0 \subseteq \mathcal{F}$ is said to be $P$*-trivial* if $P[A] \in \{0,1\}$ for all $A \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is $P$-trivial if and only if every $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$ is $P$-a.s. constant.

**Solution 1.2** Suppose that $\mathcal{F}_0$ is $P$-trivial, and consider an $\mathcal{F}_0$-measurable random variable $X : \Omega \to \mathbb{R}$. By definition we have that $\{X \leqslant a\} \in \mathcal{F}_0$ for all $a \in \mathbb{R}$, and thus $P[X \leqslant a] \in \{0,1\}$. Define

$$c := \inf\{a \in \mathbb{R} : P[X \leqslant a] = 1\}.$$

We first prove that $c \in \mathbb{R}$. Since $\{X \leqslant n\} \uparrow \{X \in \mathbb{R}\}$, then $P[X \leqslant n] \uparrow P[X \in \mathbb{R}] = 1$, and so the above infimum is over a nonempty set (i.e. $c \neq \infty$). Then, if $c = -\infty$, we have

that $P[X \leqslant -n] = 1$ for all $n \in \mathbb{N}$, and from the fact that $\{X \leqslant -n\} \downarrow \varnothing$, it follows that $1 = \lim_{n\to\infty} P[X \leqslant -n] = P[\varnothing] = 0$. We get the desired contradiction.
By the definition of the infimum, we have that $P[X \leqslant c + \frac{1}{n}] = 1$ and $P[X \leqslant c - \frac{1}{n}] = 0$ for all $n \in \mathbb{N}$. Since $\{X \leqslant c + \frac{1}{n}\} \downarrow \{X \leqslant c\}$ and $\{X \leqslant c - \frac{1}{n}\} \uparrow \{X < c\}$, we get that

$$P[X \leqslant c] = \lim_{n\to\infty} P\left[X \leqslant c + \frac{1}{n}\right] = 1, \text{ and } P[X < c] = \lim_{n\to\infty} P\left[X \leqslant c - \frac{1}{n}\right] = 0.$$

Hence, we conclude that $X = c$ $P$-a.s. because

$$P[X = c] = P[X \leqslant c] - P[X < c] = 1.$$

Conversely, suppose that every $\mathcal{F}_0$-measurable random variable is $P$-a.s. constant, and take $A \in \mathcal{F}_0$. Then,

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

is an $\mathcal{F}_0$-measurable random variable, and hence must be $P$-a.s. constant. It follows immediately that either $P[\mathbb{1}_A = 1] = P[A] = 1$ or $P[\mathbb{1}_A = 0] = P[A^c] = 1$, so that $P[A] \in \{0, 1\}$. This completes the proof.

**Exercise 1.3** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X$ an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a $\sigma$-algebra. Then, the $P$-a.s. unique random variable $Z$ such that

- $Z$ is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called the *conditional expectation of $X$ given $\mathcal{G}$* and is denoted by $E[X \,|\, \mathcal{G}]$.
[*This is the formal definition of the conditional expectation of $X$ given $\mathcal{G}$; see Section 8.2 in the lecture notes.*]

(a) Show that if $X$ is $\mathcal{G}$-measurable, then $E[X \,|\, \mathcal{G}] = X$ $P$-a.s.

(b) Show that $E[E[X \,|\, \mathcal{G}]] = E[X]$.

(c) Show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$ (that is, if $\mathcal{G}$ is $P$-trivial), then $E[X \,|\, \mathcal{G}] = E[X]$ $P$-a.s.

(d) Consider an integrable random variable $Y$ on $(\Omega, \mathcal{F}, P)$, and some constants $a, b \in \mathbb{R}$. Show that $E[aX + bY \,|\, \mathcal{G}] = aE[X \,|\, \mathcal{G}] + bE[Y \,|\, \mathcal{G}]$ $P$-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of $\Omega$, i.e., there exists a collection $(A_i)_{i=1,\ldots,n}$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \ldots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \ldots, n$. Show that

$$E[X \,|\, \mathcal{G}] = \sum_{i=1}^n E[X \,|\, A_i] \, \mathbb{1}_{A_i} \; P\text{-a.s.}$$

This says that the conditional expectation of a random variable given a finitely generated $\sigma$-algebra is a *piecewise constant* function with the constants given by the elementary conditional expectations given the sets of the generating partition.

[*This is a very useful property when one conditions on a finitely generated $\sigma$-algebra, as for instance in the multinomial model.*]

*Hint 1: Recall that $E[X \,|\, A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$ and try to write $X$ as a sum of random variables each of which only takes non-zero values on a single $A_i$.*
*Hint 2: Check that any set $A \in \mathcal{G}$ has the form $\cup_{j\in J} A_j$ for some $J \subseteq \{1, \ldots, n\}$.*

**Solution 1.3**

(a) $X$ is $\mathcal{G}$-measurable and integrable by assumption, so the first requirement in the definition of conditional expectation is satisfied for $Z = X$. Moreover, we clearly have that $E[X\mathbb{1}_A] = E[X\mathbb{1}_A]$ for all $A \in \mathcal{G}$, hence $E[X \mid \mathcal{G}] = X$ $P$-a.s.

(b) In the definition of the conditional expectation, set $A = \Omega$. Then, we obtain that $E[E[X \mid \mathcal{G}]] = E[E[X \mid \mathcal{G}]\mathbb{1}_\Omega] = E[X\mathbb{1}_\Omega] = E[X]$.

(c) Since $|E[X]| \le E[|X|]$ by Jensen's inequality and $E[|X|] < \infty$ by the assumption that $X$ is integrable, we have that $E[X]$ is integrable as well. $E[X]$ is also trivially $\mathcal{G}$-measurable since it is a constant random variable. Moreover, in this setting, $A \in \mathcal{G}$ only if $P[A] = 0$ or $P[A] = 1$. Noting that

$$\begin{aligned} E[X\mathbb{1}_A] &= 0 = E[E[X]\mathbb{1}_A], && \forall A \in \mathcal{G} \text{ such that } P[A] = 0, \\ E[X\mathbb{1}_A] &= E[X] = E[E[X]\mathbb{1}_A], && \forall A \in \mathcal{G} \text{ such that } P[A] = 1, \end{aligned}$$

we obtain $E[X \mid \mathcal{G}] = E[X]$ $P$-a.s.

(d) By the definition of the conditional expectation, we have that $E[X \mid \mathcal{G}]$ and $E[Y \mid \mathcal{G}]$ are $\mathcal{G}$-measurable and integrable; hence, the same holds for $aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$. Choosing some $A \in \mathcal{G}$, we can compute that

$$\begin{aligned} E\big[(aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}])\mathbb{1}_A\big] &= aE\big[E[X \mid \mathcal{G}]\mathbb{1}_A\big] + bE\big[E[Y \mid \mathcal{G}]\mathbb{1}_A\big] \\ &= aE[X\mathbb{1}_A] + bE[Y\mathbb{1}_A] = E[(aX + bY)\mathbb{1}_A], \end{aligned}$$

where the first equality uses the linearity of the (classical) expectation and the second uses the definition of $E[X \mid \mathcal{G}]$ and $E[Y \mid \mathcal{G}]$. By the arbitrariness of $A \in \mathcal{G}$, we can conclude that $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$ $P$-a.s.

(e) First recall that $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$. Using that

$$X = X\mathbb{1}_\Omega = X\mathbb{1}_{\cup_{i=1}^n A_i} = X\sum_{i=1}^n \mathbb{1}_{A_i} = \sum_{i=1}^n X\mathbb{1}_{A_i},$$

where the third equality holds because $A_i$ are pairwise disjoint, we get by part (d) that

$$E[X \mid \mathcal{G}] = \sum_{i=1}^n E[X\mathbb{1}_{A_i} \mid \mathcal{G}] \ \ P\text{-a.s.},$$

and hence we only have to show that $E[X\mathbb{1}_{A_i} \mid \mathcal{G}] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ $P$-a.s. for each $i \in \{1, \dots, n\}$. Since $A_i \in \mathcal{G}$ and $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i] \in \mathbb{R}$, we already know that $E[X \mid A_i]\mathbb{1}_{A_i}$ is $\mathcal{G}$-measurable and integrable. One can verify that the family of sets $A = \bigcup_{j\in J} A_j$ for $J \in 2^{\{1,\dots,n\}}$ (the power set of $\{1, \dots, n\}$) forms a $\sigma$-field. Let us denote this $\sigma$-field by $\tilde{\mathcal{G}}$. Since we clearly have $A_i \in \tilde{\mathcal{G}}$ for all $i \in \{1, \dots, n\}$, we get that $\tilde{\mathcal{G}} \supseteq \mathcal{G}$, which for any $A \in \mathcal{G}$ implies that $A = \bigcup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$. For any such $A \in \mathcal{G}$, we have that

$$\mathbb{1}_{A_i}\mathbb{1}_A = \begin{cases} \mathbb{1}_{A_i} & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

Hence, we can then compute

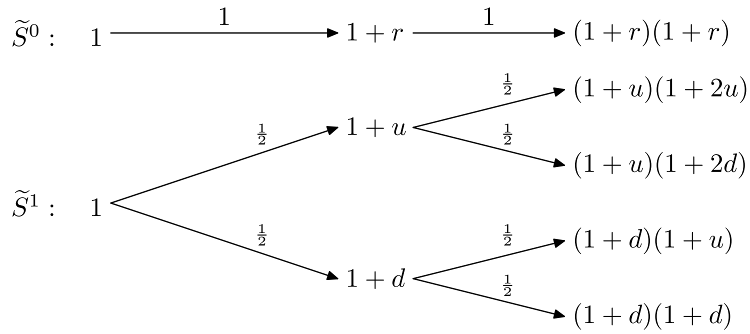
$$E\left[\left(\frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}\right)\mathbb{1}_A\right] = \begin{cases} E[X\mathbb{1}_{A_i}]\frac{P[A_i]}{P[A_i]} = E[X\mathbb{1}_{A_i}] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

On the other hand, we have that

$$E\left[X\mathbb{1}_{A_i}\mathbb{1}_A\right] = \begin{cases} E\left[X\mathbb{1}_{A_i}\right] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

This shows that $E\left[X\mathbb{1}_{A_i}\,|\,\mathcal{G}\right] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ $P$-a.s. and concludes the proof.

**Exercise 1.4** Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities:




Intuitively, this means that the volatility of $\tilde{S}^1$ increases after a stock price increase in the first period. Assume that $u$, $r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables $Y_1$ and $Y_2$ and two adapted stochastic processes $\tilde{S}^0$ and $\tilde{S}^1$ such that $\tilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.

(b) For which values of $u$ and $d$ are $Y_1$ and $Y_2$ *uncorrelated*?

(c) For which values of $u$ and $d$ are $Y_1$ and $Y_2$ *independent*?

(d) For which values of $u$, $r$ and $d$ is the discounted stock process $S^1$ a $P$-martingale?

**Solution 1.4**

(a) We construct the canonical model for this setup, a path space. Let $\Omega := \{-1, 1\}^2$, take $\mathcal{F} := 2^\Omega$ and define $P$ by

$$P\left[\{(x_1, x_2)\}\right] := p_{x_1} p_{x_1, x_2}\,,$$

where $p_1 = p_{-1} := 1/2$ and $p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := 1/2$. Next, define $Y_1$ and $Y_2$ by $Y_1((1,1)) = Y_1((1,-1)) := 1 + u$, $Y_1((-1,1)) = Y_1((-1,-1)) := 1 + d$ and $Y_2((1,1)) := 1 + 2u$, $Y_2((1,-1)) := 1 + 2d$, $Y_2((-1,1)) := 1 + u$, $Y_2((-1,-1)) := 1 + d$. Finally, define $\tilde{S}^0$ and $\tilde{S}^1$ by $\tilde{S}^0_k := (1+r)^k$ and $\tilde{S}^1_k := \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$ and set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1) = \{\emptyset, \{(1,1), (1,-1)\}, \{(-1,1), (-1,-1)\}, \Omega\}$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(b) $Y_1$ and $Y_2$ are uncorrelated if and only if $E\left[Y_1 Y_2\right] = E\left[Y_1\right] E\left[Y_2\right]$. Set $c := (u + d)/2$ to simplify the notation. Then we have

$$E\left[Y_1\right] = 1 + c \quad \text{and} \quad E\left[Y_2\right] = 1 + \frac{3}{2}c\,,$$
$$E\left[Y_1 Y_2\right] = \frac{1+u}{2}(1 + 2c) + \frac{1+d}{2}(1 + c) = (1 + c)^2 + \frac{1+u}{2}c\,.$$

Hence, we have

$$\begin{aligned} E[Y_1Y_2] - E[Y_1]\,E[Y_2] &= (1+c)^2 + \frac{1+u}{2}c - \Big((1+c)^2 + (1+c)\frac{c}{2}\Big) \\ &= (u-c)\frac{c}{2}\,. \end{aligned}$$

Since $d \leq 0 \leq u$, we have

$$(u-c)\frac{c}{2} = 0 \quad \Leftrightarrow \quad c = 0 \ \text{ or } \ u - c = 0 \quad \Leftrightarrow \quad d = -u\,.$$

In conclusion, $Y_1$ and $Y_2$ are uncorrelated if and only if $d = -u$.

(c) Since independent random variables are a fortiori uncorrelated, we only have to consider the case that $u = -d$. If $u = d = 0$, $Y_1$ and $Y_2$ are both constant and hence independent. Otherwise, if $u > 0$ we have on the one hand

$$P[Y_1 = 1+u, Y_2 = 1+u] = 0$$

and on the other hand

$$P[Y_1 = 1+u]\,P[Y_2 = 1+u] = 1/2 \cdot 1/4 = 1/8 \neq 0\,,$$

showing that in this case $Y_1$ and $Y_2$ are not independent. In conclusion, $Y_1$ and $Y_2$ are independent if and only if $u = d = 0$.

Note: If $d = -u$ and $u \neq 0$, then $Y_1$ and $Y_2$ are uncorrelated but **not** independent.

(d) $S^1$ is a $P$-martingale if and only if

$$E\big[S_1^1 \,\big|\, \mathcal{F}_0\big] = S_0^1 \quad P\text{-a.s.} \quad \text{and} \quad E\big[S_2^1 \,\big|\, \mathcal{F}_1\big] = S_1^1 \quad P\text{-a.s.} \tag{1}$$

If $u = d = 0$, it is straightforward to check that $S^1$ is a $P$-martingale if and only if $r = 0$. Next, assume that $u > d$. Since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and $Y_1 > 0$, (1) is equivalent to

$$E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 \,|\, Y_1] = 1 + r \quad P\text{-a.s.}$$

Since $Y_1$ only takes two values, this is equivalent to

$$E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 \,|\, Y_1 = 1+u] = 1 + r \quad \text{and} \quad E[Y_2 \,|\, Y_1 = 1+d] = 1 + r\,.$$

This is equivalent to

$$\begin{aligned} 1 + (u+d)/2 &= 1 + r\,, \\ 1 + u + d &= 1 + r\,, \\ 1 + (u+d)/2 &= 1 + r\,. \end{aligned}$$

Subtracting the first from the second equation yields $(u+d)/2 = 0$, which in turn implies $r = 0$. In conclusion, $S^1$ is a $P$-martingale if and only if $r = 0$ and $d = -u$.