Coordinator
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Mathematical Foundations for Finance

Exercise Sheet 2

Please hand in your solutions by 12:00 on Wednesday, October 9 via the course homepage.

Exercise 2.1 Consider a probability space $(\Omega, \mathcal{F}, P)$. Fix a finite time horizon $T \in \mathbb{N}$, and let $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ be random variables. For $k = 0, \dots, T$, define

$$\widetilde{S}_k^0 := \prod_{j=1}^{k} (1 + r_j), \qquad \widetilde{S}_k^1 := S_0^1 \prod_{j=1}^{k} Y_j,$$

where $S_0^1 > 0$ is some constant.

(a) Consider the filtration $\mathbb{F}' = (\mathcal{F}'_k)_{k=0,\dots,T}$ generated by $Y = (Y_k)_{k=1,\dots,T}$ and $r = (r_k)_{k=1,\dots,T}$, so that

$$\mathcal{F}'_0 = \{\varnothing, \Omega\},$$
$$\mathcal{F}'_k = \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k), \quad k = 1, \dots, T.$$

Show that if r is $\mathbb{F}'$-predictable, then $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\widetilde{S}_0^1, \widetilde{S}_1^1, \dots, \widetilde{S}_k^1)$ for all $k = 0, \dots, T$.

(b) Recall that a strategy $\varphi = (\varphi^0, \vartheta)$ is *self-financing* if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing does not depend on discounting. That is, if $D = (D_k)_{k=0,\dots,T}$ is any positive adapted process and $\bar{S}_k^i := S_k^i D_k$ for each $k = 0, \dots, T$ and $i = 0, 1$, then the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\bar{C}(\varphi)$, determined by

$$\Delta \bar{C}_{k+1}(\varphi) := (\varphi_{k+1}^0 - \varphi_k^0)\bar{S}_k^0 + (\vartheta_{k+1} - \vartheta_k)\bar{S}_k^1,$$

is constant over time.

(c) Show that the notion of self-financing is numéraire-invariant, i.e. it does not matter if the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

Exercise 2.2 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and let $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ be stopping times.

(a) Show that $\tau \wedge \sigma := \min\{\tau, \sigma\}$ is a stopping time.

(b) Show that $\tau \vee \sigma := \max\{\tau, \sigma\}$ is a stopping time.

(c) Show that a function $\rho : \Omega \to \mathbb{N} \cup \{\infty\}$ is an $\mathbb{F}$-stopping time if and only if $\{\rho = k\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$.

(d) Show that $\tau + \sigma$ is a stopping time.

(e) Suppose $\tau \geqslant \sigma$. Is $\tau - \sigma$ a stopping time?

(f) Suppose that $X = (X_k)_{k\in\mathbb{N}}$ is an adapted $\mathbb{R}^d$-valued process, and let $a \in \mathbb{R}$. Show that

$$\rho := \inf\{k : |X_k| \geqslant a\}$$

is a stopping time.

Show that ρ is still a stopping time if "$\geqslant$" is replaced by any of "$>$", "$\leqslant$" or "$<$".

Exercise 2.3 Fix a probability space $(\Omega, \mathcal{F}, P)$ and a finite time horizon $T \geqslant 2$. Consider a market (S^0, S^1) consisting of a bank account and a stock, respectively. Assume that $S^0 \equiv 1$, $S^1_0 = 1$ and $S^1_k > 0$ for all $k = 1, \dots, T$. Fix $0 < \ell < 1 < u$, and define the maps $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ by

$$\tau(\omega) := \inf\{k = 0, \dots, T : S^1_k(\omega) \leqslant \ell\} \wedge T,$$
$$\sigma(\omega) := \inf\{k = \tau(\omega), \dots, T : S^1_k(\omega) \geqslant u\} \wedge T.$$

We use here the standard convention $\inf \varnothing = +\infty$.

(a) Define the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T}$ on $(\Omega, \mathcal{F})$ by $\mathcal{F}_k := \sigma(S^1_i : 0 \leqslant i \leqslant k)$. Show that τ and σ are $\mathbb{F}$-stopping times.

(b) Define the process $\vartheta = (\vartheta_k)_{k=1,\dots,T}$ by

$$\vartheta_k := \mathbb{1}_{\{\tau < k \leqslant \sigma\}}, \ k = 1, \dots, T.$$

Show that ϑ is $\mathbb{F}$-predictable and $\vartheta_1 = 0$.

(c) Construct φ^0 such that the strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing with $V_0(\varphi) = 0$, and derive a formula for the discounted value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times τ and σ.

(d) Describe the trading strategy φ in words.

Exercise 2.4 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model.* More precisely, the price processes of the assets are defined by

$$\begin{aligned} \widetilde{S}^0_k &= (1+r)^k \qquad \text{for } k = 0, 1, \dots, T, \\ \frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k} &= Y_{k+1} \qquad \text{for } k = 0, 1, \dots, T-1, \end{aligned}$$

where the Y_k are i.i.d., taking values $1+u$ with probability $p \in (0,1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d > -1$ and $r > -1$.

(a) Suppose that $r \leq d$. Show that in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

(b) Suppose that $r \geq u$. Show that also in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*