# Mathematical Foundations for Finance

## Exercise Sheet 2

**Exercise 2.1** Consider a probability space $(\Omega, \mathcal{F}, P)$. Fix a finite time horizon $T \in \mathbb{N}$, and let $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ be random variables. For $k = 0, \dots, T$, define

$$\widetilde{S}_k^0 := \prod_{j=1}^{k}(1 + r_j), \qquad \widetilde{S}_k^1 := S_0^1 \prod_{j=1}^{k} Y_j,$$

where $S_0^1 > 0$ is some constant.

(a) Consider the filtration $\mathbb{F}' = (\mathcal{F}_k')_{k=0,\dots,T}$ generated by $Y = (Y_k)_{k=1,\dots,T}$ and $r = (r_k)_{k=1,\dots,T}$, so that

$$\begin{aligned} &\mathcal{F}_0' = \{\varnothing, \Omega\}, \\ &\mathcal{F}_k' = \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k), \quad k = 1, \dots, T. \end{aligned}$$

Show that if $r$ is $\mathbb{F}'$-predictable, then $\mathcal{F}_k' = \mathcal{F}_k := \sigma(\widetilde{S}_0^1, \widetilde{S}_1^1, \dots, \widetilde{S}_k^1)$ for all $k = 0, \dots, T$.

(b) Recall that a strategy $\varphi = (\varphi^0, \vartheta)$ is *self-financing* if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing does not depend on discounting. That is, if $D = (D_k)_{k=0,\dots,T}$ is any positive adapted process and $\bar{S}_k^i := S_k^i D_k$ for each $k = 0, \dots, T$ and $i = 0, 1$, then the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\bar{C}(\varphi)$, determined by

$$\Delta \bar{C}_{k+1}(\varphi) := (\varphi_{k+1}^0 - \varphi_k^0)\bar{S}_k^0 + (\vartheta_{k+1} - \vartheta_k)\bar{S}_k^1,$$

is constant over time.

(c) Show that the notion of self-financing is numéraire-invariant, i.e. it does not matter if the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

**Solution 2.1**

(a) The proof is by induction on $k$. Since $\widetilde{S}^1_0 = S^1_0$ is constant, then

$$\mathcal{F}_0 = \sigma(\widetilde{S}^1_0) = \{\varnothing, \Omega\} = \mathcal{F}'_0.$$

Now assume that $\mathcal{F}_k = \mathcal{F}'_k$ for some $k \geqslant 0$. We need to show that $\mathcal{F}_{k+1} = \mathcal{F}'_{k+1}$. To this end, note that

$$\widetilde{S}^1_{k+1} = \widetilde{S}^1_k Y_{k+1},$$

which is $\mathcal{F}'_{k+1}$-measurable, since $\widetilde{S}^1_k$ (because $\mathcal{F}'_k = \mathcal{F}_k$) and $Y_{k+1}$ are. So since $\widetilde{S}^1_j$ is $\mathcal{F}'_{k+1}$-measurable for all $0 \leqslant j \leqslant k+1$, we have $\mathcal{F}_{k+1} \subseteq \mathcal{F}'_{k+1}$. Conversely, writing

$$Y_{k+1} = \frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k}$$

(which is well-defined since $\widetilde{S}^1_k > 0$), we see that $Y_{k+1}$ is $\mathcal{F}_{k+1}$-measurable. Also, since $r$ is $\mathbb{F}'$-predictable, then $r_{k+1}$ is $\mathcal{F}_k$-measurable (because $\mathcal{F}_k = \mathcal{F}'_k$). By the same reasoning as above, since $Y_j$ and $r_j$ are $\mathcal{F}_{k+1}$-measurable for all $0 \leqslant j \leqslant k+1$, we get $\mathcal{F}'_{k+1} \subseteq \mathcal{F}_{k+1}$. Hence, $\mathcal{F}'_{k+1} = \mathcal{F}_{k+1}$. By induction, this completes the proof.

(b) By applying the definition of the incremental cost $k$ times, we get

$$C_k(\varphi) = \Delta C_k(\varphi) + C_{k-1}(\varphi) = C_0(\varphi) + \sum_{j=1}^{k} \Delta C_j(\varphi).$$

It follows that the cost process $C(\varphi)$ is constant over time (and equal to the initial investment $\varphi^0_0$ in the bank account) if and only if we have that $\Delta C_k(\varphi) = 0$ for all $k$. By the definition of $\Delta C_k(\varphi)$, this equality reads

$$(\varphi^0_k - \varphi^0_{k-1}) S^0_{k-1} + (\vartheta_k - \vartheta_{k-1}) S^1_{k-1} = 0, \quad \forall k.$$

Multiplying both sides of the equation by $D_{k-1}$, we obtain the same condition for the prices $\bar{S}$:

$$\Delta \bar{C}_k(\varphi) = (\varphi^0_k - \varphi^0_{k-1}) \bar{S}^0_{k-1} + (\vartheta_k - \vartheta_{k-1}) \bar{S}^1_{k-1} = 0.$$

Since we also have the identity

$$\bar{C}_k(\varphi) = \bar{C}_0(\varphi) + \sum_{j=1}^{k} \Delta \bar{C}_j(\varphi),$$

it follows that the undiscounted cost process $\overline{C}(\phi)$ is constant over time. The other direction is established in the same way, thus completing the proof.

(c) This follows immediately from part (b) by first setting $D = \widetilde{S}^1 / \widetilde{S}^0$ and then $D = \widetilde{S}^0 / \widetilde{S}^1$.

**Exercise 2.2** Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and let $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ be stopping times.

(a) Show that $\tau \wedge \sigma := \min\{\tau, \sigma\}$ is a stopping time.

(b) Show that $\tau \vee \sigma := \max\{\tau, \sigma\}$ is a stopping time.

(c) Show that a function $\rho : \Omega \to \mathbb{N} \cup \{\infty\}$ is an $\mathbb{F}$-stopping time if and only if $\{\rho = k\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$.

(d) Show that $\tau + \sigma$ is a stopping time.

(e) Suppose $\tau \geqslant \sigma$. Is $\tau - \sigma$ a stopping time?

(f) Suppose that $X = (X_k)_{k \in \mathbb{N}}$ is an adapted $\mathbb{R}^d$-valued process, and let $a \in \mathbb{R}$. Show that

$$\rho := \inf\{k : |X_k| \geqslant a\}$$

is a stopping time.

Show that $\rho$ is still a stopping time if "$\geqslant$" is replaced by any of "$>$", "$\leqslant$" or "$<$".

**Solution 2.2**

(a) We have $\{\tau \wedge \sigma \leqslant k\} = \{\tau \leqslant k\} \cup \{\sigma \leqslant k\} \in \mathcal{F}_k$, since $\tau$ and $\sigma$ are stopping times.

(b) We have $\{\tau \vee \sigma \leqslant k\} = \{\tau \leqslant k\} \cap \{\sigma \leqslant k\} \in \mathcal{F}_k$, since $\tau$ and $\sigma$ are stopping times.

(c) If $\tau$ is a stopping time, then $\{\tau = k\} = \{\tau \leqslant k\} \setminus \{\tau \leqslant k-1\} \in \mathcal{F}_k$, as needed. For the converse, we note that $\{\tau \leqslant k\} = \cup_{j=0}^{k} \{\tau = j\} \in \mathcal{F}_k$, as required.

(d) We have

$$\{\tau + \sigma = k\} = \bigcup_{j=0}^{k} \{\tau = j\} \cap \{\sigma = k - j\}.$$

By part (c), we can conclude that $\{\tau + \sigma = k\} \in \mathcal{F}_k$. Then again by part (c), this implies that $\tau + \sigma$ is a stopping time.

(e) No. Keeping part (c) in mind, take stopping times $\tau \equiv 1$ and $\sigma : \Omega \to \{0, 1\}$, where $\{\sigma = 1\} \in \mathcal{F}_1 \setminus \mathcal{F}_0$. Then we have

$$\{\tau - \sigma = 0\} = \{\sigma = 1\} \notin \mathcal{F}_0,$$

so that $\tau - \sigma$ cannot be a stopping time.

(f) For each $n \in \mathbb{N}$, we have

$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| \geqslant a\}.$$

Since $X$ is adapted, then for all $k = 0, \dots, n$, we have

$$\{|X_k| \geqslant a\} \in \mathcal{F}_k \subseteq \mathcal{F}_n.$$

Since a $\sigma$-field is closed under finite unions, it follows that $\{\rho \leqslant n\} \in \mathcal{F}_n$, and hence $\rho$ is a stopping time.

Similarly, when "$\geqslant$" is replaced by "$>$", "$\leqslant$" or "$<$", we have the following equalities, respectively:

$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| > a\}, \qquad \{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| \leqslant a\},$$
$$\{\rho \leqslant n\} = \bigcup_{k=0}^{n} \{|X_k| < a\}.$$

By the same reasoning as above, we can conclude that $\rho$ is still a stopping time in these cases.

**Exercise 2.3** Fix a probability space $(\Omega, \mathcal{F}, P)$ and a finite time horizon $T \geqslant 2$. Consider a market $(S^0, S^1)$ consisting of a bank account and a stock, respectively. Assume that $S^0 \equiv 1$, $S^1_0 = 1$ and $S^1_k > 0$ for all $k = 1, \dots, T$. Fix $0 < \ell < 1 < u$, and define the maps $\tau, \sigma : \Omega \to \mathbb{N} \cup \{\infty\}$ by

$$\tau(\omega) := \inf\{k = 0, \dots, T : S^1_k(\omega) \leqslant \ell\} \wedge T,$$
$$\sigma(\omega) := \inf\{k = \tau(\omega), \dots, T : S^1_k(\omega) \geqslant u\} \wedge T.$$

We use here the standard convention $\inf \varnothing = +\infty$.

(a) Define the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T}$ on $(\Omega, \mathcal{F})$ by $\mathcal{F}_k := \sigma(S^1_i : 0 \leqslant i \leqslant k)$. Show that $\tau$ and $\sigma$ are $\mathbb{F}$-stopping times.

(b) Define the process $\vartheta = (\vartheta_k)_{k=1,\dots,T}$ by

$$\vartheta_k := \mathbb{1}_{\{\tau < k \leqslant \sigma\}}, \ k = 1, \dots, T.$$

Show that $\vartheta$ is $\mathbb{F}$-predictable and $\vartheta_1 = 0$.

(c) Construct $\varphi^0$ such that the strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing with $V_0(\varphi) = 0$, and derive a formula for the discounted value process $V(\varphi)$ involving only the discounted stock price $S^1$ and the stopping times $\tau$ and $\sigma$.

(d) Describe the trading strategy $\varphi$ in words.

**Solution 2.3**

(a) Since $S^1$ is adapted to $\mathbb{F}$ and a non-negative integer (in our case, $T$) is a stopping time, we can use Exercise 2.2(f) and Exercise 2.2(a) to conclude that $\tau$ is a stopping time.

To prove that $\sigma$ is a stopping time, we observe that $\{\tau \leq k\} = \varnothing \in \mathcal{F}_k$ for $k = 0, 1$, and $\{\tau \leq T\} = \Omega \in \mathcal{F}_T$. For $k = 2, \ldots, T-1$, we have

$$\{\sigma \leq k\} = \bigcup_{1 \leq i < j \leq k} \{S_i^1 \leq \ell, S_j^1 \geq u\} \in \mathcal{F}_k$$

because $\{S_i^1 \leq \ell, S_j^1 \geq u\} = \{S_i^1 \leq \ell\} \cap \{S_j^1 \geq u\} \in \mathcal{F}_j \subset \mathcal{F}_k$.

(b) Since an indicator function is measurable if and only if the indicating set belongs to the $\sigma$-field, we need to show that $\{\tau < k \leqslant \sigma\} \in \mathcal{F}_{k-1}$ for each $k = 1, \ldots, T$. To this end, we write

$$\{\tau < k \leqslant \sigma\} = \{\tau < k\} \cap \{\sigma \geqslant k\} = \{\tau \leqslant k-1\} \cap \{\sigma \leqslant k-1\}^c.$$

Since $\tau$ and $\sigma$ are stopping times, it follows that the above set belongs to $\mathcal{F}_{k-1}$, completing the proof.

Finally, note that since $S_0^1 = 1$, we must have $\tau \geqslant 1$, so that

$$\vartheta_1 = \mathbb{1}_{\{\tau < 1 \leqslant \sigma\}} = \mathbb{1}_{\varnothing} = 0,$$

as required.

(c) Recall that a strategy $\varphi = (\phi^0, \vartheta)$ is *self-financing* if $C_k(\varphi) = C_0(\varphi)$ for all $k$. By definition, $C_k(\varphi) = V_k(\varphi) - G_k(\vartheta)$, and since $C_0(\varphi) = V_0(\varphi)$, we may rewrite this condition as $V_k(\varphi) = V_0(\varphi) + G_k(\vartheta)$ for all $k$.

For our setting, we compute

$$\begin{aligned} G_k(\vartheta) &= \sum_{j=1}^{k} \vartheta_j \Delta S_j^1 = \sum_{j=1}^{k} \mathbb{1}_{\{\tau < j \leqslant \sigma\}} \Delta S_j^1 \\ &= \sum_{j=1}^{k} \mathbb{1}_{\{\tau < j \leqslant \sigma\}} (S_j^1 - S_{j-1}^1) = \sum_{j=k\wedge\tau+1}^{k\wedge\sigma} (S_j^1 - S_{j-1}^1) \\ &= S_{k\wedge\sigma}^1 - S_{k\wedge\tau}^1. \end{aligned}$$

By definition, $V_k(\varphi) = \varphi_k^0 + \vartheta_k S_k^1 = \varphi_k^0 + \mathbb{1}_{\{\tau<k\leqslant\sigma\}} S_k^1$, and thus $\varphi$ is self-financing with $V_0(\varphi) = 0$ if and only if $\varphi_0^0 = 0$ and

$$\varphi_k^0 + \mathbb{1}_{\{\tau<k\leqslant\sigma\}} S_k^1 = S_{k\wedge\sigma}^1 - S_{k\wedge\tau}^1, \ \forall k = 1, \ldots, T.$$

We thus take $\varphi_0^0 = 0$, and for all $k = 1, \ldots, T$,

$$\begin{aligned} \varphi_k^0 &= S_{k\wedge\sigma}^1 - S_{k\wedge\tau}^1 - \mathbb{1}_{\{\tau<k\leqslant\sigma\}} S_k^1 \\ &= -S_\tau^1 \mathbb{1}_{\{\tau<k\leqslant\sigma\}} + \mathbb{1}_{\{\sigma<k\}} (S_\sigma^1 - S_\tau^1). \end{aligned}$$

Finally, since $C_k(\varphi) = C_0(\varphi) = 0$ for all $k$, we have

$$V_k(\varphi) = G_k(\vartheta) = S^1_{k\wedge\sigma} - S^1_{k\wedge\tau},$$

so that

$$V(\varphi) = (S^1)^\sigma - (S^1)^\tau.$$

(d) This strategy can be described as a "buy low and sell high" strategy. When the discounted price of the stock falls below $\ell$, one borrows money to buy one share of the stock. As soon as the discounted price of the stock climbs above $u$, one sells the share, pays back the loan and stores the remaining money in the bank account.

**Exercise 2.4** Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model*. More precisely, the price processes of the assets are defined by

$$\begin{aligned} \widetilde{S}^0_k &= (1+r)^k && \text{for } k = 0, 1, \dots, T, \\ \frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k} &= Y_{k+1} && \text{for } k = 0, 1, \dots, T-1, \end{aligned}$$

where the $Y_k$ are i.i.d., taking values $1+u$ with probability $p \in (0,1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d > -1$ and $r > -1$.

(a) Suppose that $r \leq d$. Show that in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

(b) Suppose that $r \geq u$. Show that also in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

**Solution 2.4**

(a) If $r \leq d$, the stock grows in each state of the world and in all trading periods at least as fast as the bank account, but in some states of the world faster since $u > d$. In mathematical terms, this means that for $k = 1, \dots, T$, we have

$$Y_k \geq 1 + r \quad P\text{-a.s.} \quad \text{and} \quad P[Y_k > 1 + r] > 0\,.$$

In terms of the discounted stock price $S^1$, this means that for $k = 1, \dots, T$, we have

$$S^1_k \geq S^1_{k-1} \quad P\text{-a.s.} \quad \text{and} \quad P\left[S^1_k > S^1_{k-1}\right] > 0\,. \tag{1}$$

Therefore, the obvious arbitrage opportunity consists in borrowing money at time 0 from the bank account to buy, say, one stock and holding the stock until the time horizon $T$. In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where $\vartheta$ is given by $\vartheta_0 = 0$ and $\vartheta_k = 1$, $k = 1, \dots,$

$T$, which is deterministic and therefore a fortiori predictable. Moreover, by formula (1.2.8) in the lecture notes, for $k = 1, \ldots, T$, we have

$$V_k(\varphi) = G_k(\vartheta) = \sum_{j=1}^{k}(1 \times \Delta S_j) = S_k^1 - S_0^1 \quad P\text{-a.s.}$$

Hence, by (1) we may deduce on the one hand that $V(\varphi) \geq 0$, whence $\varphi$ is 0-admissible and a fortiori admissible, and on the other hand that $P\left[V_T(\varphi) > 0\right] > 0$, whence $\varphi$ is an arbitrage opportunity.

(b) If $r \geq u$, the stock grows in each state of the world and in all trading periods at most as fast as the bank account, but in some states of the world more slowly since $u > d$. In mathematical terms, this means that for $k = 1, \ldots, T$, we have

$$Y_k \leq 1 + r \quad P\text{-a.s.} \quad \text{and} \quad P\left[Y_k < 1 + r\right] > 0\,.$$

In terms of the discounted stock price $S^1$, this means that for $k = 1, \ldots, T$, we have

$$S_k^1 \leq S_{k-1}^1 \quad P\text{-a.s.} \quad \text{and} \quad P\left[S_k^1 < S_{k-1}^1\right] > 0\,. \tag{2}$$

Therefore, the obvious arbitrage opportunity consists in short-selling, say, one stock at time 0 and investing the money into the bank account until the time horizon $T$. In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where $\vartheta$ is given by $\vartheta_0 = 0$ and $\vartheta_k = -1$, $k = 1, \ldots, T$, which is deterministic and therefore a fortiori predictable. Moreover, by formula (2.8) in the lecture notes, for $k = 1, \ldots, T$, we have

$$V_k(\varphi) = G_k(\vartheta) = \sum_{j=1}^{k}(-1 \times \Delta S_j) = S_0^1 - S_k^1 \quad P\text{-a.s.}$$

Hence, by (2) we may deduce on the one hand that $V(\varphi) \geq 0$, whence $\varphi$ is 0-admissible and a fortiori admissible, and on the other hand that $P\left[V_T(\varphi) > 0\right] > 0$, whence $\varphi$ is an arbitrage opportunity.