Mathematical Foundations for Finance

Exercise Sheet 3

Please hand in your solutions by 12:00 on Wednesday, October 16 via the course homepage.

Exercise 3.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0}^T$ with $\mathcal{F}_0$ trivial. Let $X = (X_k)_{k=0}^T$ be a supermartingale. Show that $X_0 \geq E[X_T]$ always, and that we have $X_0 = E[X_T]$ if and only if X is a martingale.

Exercise 3.2 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$.

(a) Let X be a martingale. Show that for any bounded and convex function $f\colon \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k\in\mathbb{N}_0}$ is a submartingale.

Could we replace the request of f being bounded with a more general condition?

Hint: You may use that finite-valued convex functions are continuous.

(b) Let X be a submartingale, and let $\vartheta = (\vartheta_k)_{k\in\mathbb{N}_0}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let X be a submartingale and let τ be a stopping time. Show that the stopped process $X^\tau = (X_k^\tau)_{k\in\mathbb{N}_0}$ defined by $X_k^\tau = X_{k\wedge\tau}$ is a submartingale.

Exercise 3.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *trinomial model.* This is like a binomial model a special case of a *multinomial model*, and the distribution of Y_k under P is given by

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1 \\ 1+m & \text{with probability } p_2 \\ 1+u & \text{with probability } p_3 \end{cases}$$

where p_1, p_2, $p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d, m and u are mnemonics for *down*, *middle* and *up*.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$. For $T = 1$, consider an arbitrary self-financing strategy $\varphi \widehat{=} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is nonnegative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.

Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a *probability vector* $(q_1, q_2, q_3) \in (0,1)^3$, where $q_k = Q\,[Y_1 = 1 + y_k]$, $k = 1$, 2, 3, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$. (A *probability vector in* $\mathbb{R}^n$, $n \in \mathbb{N}$ *is a nonnegative vector in* $\mathbb{R}^n$ *whose coordinates sum up to 1.*)

(c) Assume now that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$. For $T = 2$, give a parametrisation of all *equivalent martingale measures* (EMMs) for S^1.

Hint: A probability measure Q equivalent to P on $\mathcal{F}_2$ can be uniquely described by four *probability vectors* (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1$, 2, 3, where $q_j = Q\,[Y_1 = 1 + y_j]$ and $q_{j,k} = Q\,[Y_2 = 1 + y_k \,|\, Y_1 = 1 + y_j]$, $j, k = 1$, 2, 3, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.