Mathematical Foundations for Finance

Exercise Sheet 3

Exercise 3.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0}^T$ with $\mathcal{F}_0$ trivial. Let $X = (X_k)_{k=0}^T$ be a supermartingale. Show that $X_0 \geq E[X_T]$ always, and that we have $X_0 = E[X_T]$ if and only if X is a martingale.

Solution 3.1 The process X is a supermartingale, so $E[X_T \mid \mathcal{F}_0] \leq X_0$, and since $\mathcal{F}_0$ is trivial $E[X_k] = E[X_k \mid \mathcal{F}_0] \leq X_0$. So $E[X_T] \leq E[X_k] \leq E[X_0]$ and $E[X_T - X_k \mid F_k] \leq 0$ has expectation $E[X_T] - E[X_k]$. If X is a martingale, we have equality everywhere and hence $E[X_T] = E[X_0]$. Conversely, if $E[X_T] = E[X_0]$, then $E[X_T] \leq E[X_k] = E[X_0]$ implies $E[X_T] = E[X_k]$; so the nonpositive random variable $E[X_T - X_k \mid \mathcal{F}_k]$ has expectation zero and hence must be zero P-a.s.. This gives $E[X_T - X_k \mid \mathcal{F}_k] = 0$ P-a.s. and so X is a martingale.

Exercise 3.2 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$.

(a) Let X be a martingale. Show that for any bounded and convex function $f : \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k \in \mathbb{N}_0}$ is a submartingale.

Could we replace the request of f being bounded with a more general condition?

Hint: You may use that finite-valued convex functions are continuous.

(b) Let X be a submartingale, and let $\vartheta = (\vartheta_k)_{k \in \mathbb{N}_0}$ be a bounded, nonnegative and predictable process. Show that the stochastic integral process $\vartheta \bullet X$, defined by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j = \sum_{j=1}^{k} \vartheta_j (X_j - X_{j-1}),$$

is a submartingale.

Conclude that $E[\vartheta \bullet X_k] \geqslant 0$ for all $k \in \mathbb{N}_0$.

(c) Let X be a submartingale and let τ be a stopping time. Show that the stopped process $X^\tau = (X_k^\tau)_{k \in \mathbb{N}_0}$ defined by $X_k^\tau = X_{k \wedge \tau}$ is a submartingale.

Solution 3.2

(a) The process $f(X)$ is integrable because f is bounded. Since X is adapted (because it is a martingale) and f is continuous (since it is finite-valued and convex), it follows that $f(X)$ is adapted. It remains to show the submartingale inequality. For $0 \leqslant m < n$, we write

$$E[f(X_n) \mid \mathcal{F}_m] \geqslant f(E[X_n \mid \mathcal{F}_m]) = f(X_m),$$

where the first step used the (conditional) Jensen's inequality, and the second step the martingale property. This concludes the proof.

A look at the proof shows that if we replace the condition "f is bounded" by "$f(X)$ is integrable", the result still holds.

(b) Since ϑ is predictable and X is adapted, then $\vartheta_j(X_j - X_{j-1})$ is $\mathcal{F}_j$-measurable for all $j \in \mathbb{N}$. It follows that $\vartheta \bullet X_k$ is $\mathcal{F}_k$-measurable, so that $\vartheta \bullet X$ is adapted. Also, since ϑ is bounded and X is integrable, we have that $\vartheta \bullet X$ is integrable. It remains to establish the submartingale inequality. Note that it suffices to show

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0, \ \forall k \in \mathbb{N}_0.$$

To this end, we write

$$\begin{aligned} E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] &= E[\vartheta_{k+1}(X_{k+1} - X_k) \mid \mathcal{F}_k] \\ &= \vartheta_{k+1} E[X_{k+1} - X_k \mid \mathcal{F}_k], \end{aligned}$$

where in the last step we used that ϑ_{k+1} is $\mathcal{F}_k$-measurable and bounded. Since X is a submartingale, then $E[X_{k+1} - X_k \mid \mathcal{F}_k] \geqslant 0$. Since also ϑ_{k+1} is nonnegative by assumption, we have

$$E[\vartheta \bullet X_{k+1} - \vartheta \bullet X_k \mid \mathcal{F}_k] \geqslant 0,$$

as required.

Since $\vartheta \bullet X$ is a submartingale null at zero, we have for all $k \in \mathbb{N}_0$ that

$$E[\vartheta \bullet X_k] = E\Big[E[\vartheta \bullet X_k \mid \mathcal{F}_0]\Big] \geqslant E[\vartheta \bullet X_0] = 0.$$

(c) For $k \in \mathbb{N}_0$, we have

$$X_k^\tau = X_{k \wedge \tau} = X_0 + \sum_{j=1}^{k \wedge \tau} (X_j - X_{j-1}) = X_0 + \sum_{j=1}^{k} \mathbb{1}_{\{\tau \geqslant j\}} (X_j - X_{j-1}).$$

So if we set $\vartheta = (\vartheta_k)_{k \in \mathbb{N}}$ with $\vartheta_k := \mathbb{1}_{\{\tau \geqslant k\}}$, then

$$X_k^\tau = X_0 + \vartheta \bullet X_k, \ \forall k \in \mathbb{N}_0.$$

Since τ is a stopping time, then ϑ is a predictable process. Since ϑ is also bounded and nonnegative, and X is a submartingale, we may apply part (b) to conclude that $\vartheta \bullet X$ is a submartingale. Also, note that because X_0 is $\mathcal{F}_0$-measurable and integrable, then the process $(X_0)_{k \in \mathbb{N}_0}$ is a submartingale (in fact a martingale). Since the sum of two submartingales is a submartingale, we can conclude that X^τ is a submartingale, as required.

Exercise 3.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *trinomial model.* This is like a binomial model a special case of a *multinomial model*, and the distribution of Y_k under P is given by

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1 \\ 1+m & \text{with probability } p_2 \\ 1+u & \text{with probability } p_3 \end{cases}$$

where p_1, p_2, $p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d, m and u are mnemonics for *down*, *middle* and *up*.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$. For $T = 1$, consider an arbitrary self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is nonnegative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.

Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a *probability vector* $(q_1, q_2, q_3) \in (0,1)^3$, where $q_k = Q\left[Y_1 = 1 + y_k\right]$, $k = 1$, 2, 3, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$. (A *probability vector in* $\mathbb{R}^n$*,* $n \in \mathbb{N}$ *is a nonnegative vector in* $\mathbb{R}^n$ *whose coordinates sum up to 1.*)

(c) Assume now that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$. For $T = 2$, give a parametrisation of all *equivalent martingale measures* (EMMs) for S^1.

Hint: A probability measure Q equivalent to P on $\mathcal{F}_2$ can be uniquely described by four *probability vectors* (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1$, 2, 3, where $q_j = Q\left[Y_1 = 1 + y_j\right]$ and $q_{j,k} = Q\left[Y_2 = 1 + y_k \,|\, Y_1 = 1 + y_j\right]$, $j, k = 1$, 2, 3, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

Solution 3.3

(a) Let us compute the total gain $G_1(\theta)$ at time $T = 1$:

$$G_1(\theta) = \theta_1^1 \Delta S_1^1 = \theta_1^1 (S_1^1 - S_0^1) = \theta_1^1 S_0^1 \left(\frac{Y_1}{1+r} - 1 \right) = \theta_1^1 S_0^1 \times \begin{cases} \frac{u-r}{1+r} & \text{with probability } p_3, \\ \frac{m-r}{1+r} & \text{with probability } p_2, \\ \frac{d-r}{1+r} & \text{with probability } p_1. \end{cases}$$

Recall that $u - r = 0.25 > 0$ and $d - r = -0.5 < 0$. Hence $P[G_1(\theta) \geq 0] = 1$ if and only if $\theta_1^1 S_0^1 = 0$. As a result, we can conclude that

$$P[G_1(\theta) \geq 0] = 1 \quad \Leftrightarrow \quad \theta_1^1 = 0 \quad \Leftrightarrow \quad P[G_1(\theta) = 0] = 1.$$

Assume now that $V_0 = 0$ and note that in this case $V_1(\varphi) = G_1(\theta)$. The above argument proves that if $V_1(\varphi) \geq 0$ P-a.s., then $V_1(\varphi) = 0$ P-a.s., and by Proposition 1.1 3) in the lecture notes, we know that this is equivalent to saying that S^1 is arbitrage-free.

(b) Let $(q_1, q_2, q_3) \in (0,1)^3$ be a probability vector and Q be defined by

$$Q\left[Y_1 = 1 + y_k\right] := q_k, \quad k = 1, 2, 3,$$

where $y_1 := d$, $y_2 := m$ and $y_3 := u$. Then S^1 is a Q-martingale if and only if S^1 is adapted to the considered filtration (take the filtration generated by S^1 itself), integrable (the probability space is finite here, so all random variables are integrable), and

$$\begin{aligned} E_Q\left[S_1^1\right] = S_0^1 \quad &\Leftrightarrow \quad E_Q\left[S_0^1 Y_1/(1+r)\right] = S_0^1 \quad \Leftrightarrow \quad E_Q\left[Y_1\right] = 1 + r \\ &\Leftrightarrow \quad q_1 \times (1+d) + q_2 \times (1+m) + q_3 \times (1+u) = 1 + r \\ &\Leftrightarrow \quad q_1 \times d + q_2 \times m + q_3 \times u = r \\ &\Leftrightarrow \quad -0.5 q_1 + 0 q_2 + 0.25 q_3 = 0 \\ &\Leftrightarrow \quad q_3 = 2 q_1 \,. \end{aligned}$$

Recall that in order to make Q a probability measure, we need to have $q_1 + q_2 + q_3 = 1$; hence choosing $q_1 = 0.25$, we obtain that $q_3 = 0.5$ and $q_2 = 0.25$. Noting that $q_1, q_2, q_3 \in (0,1)$, we can also observe that Q is a probability measure equivalent to P and thus an EMM for S^1.

(c) Let $(q_1, q_2, q_3), (q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1, 2, 3$, be probability vectors and $Q \approx P$ on $\mathcal{F}_2 = \sigma(Y_1, Y_2)$ be defined by

$$Q\left[Y_2 = 1 + y_k, Y_1 = 1 + y_j\right] := q_j q_{j,k}, \quad j, k = 1, 2, 3,$$

where $y_1 := d$, $y_2 := m$ and $y_3 := u$. Then S^1 is a Q-martingale if and only if it is adapted, integrable and

$$E_Q\left[S_1^1\right] = S_0^1 \quad \text{and} \quad E_Q\left[S_2^1 \,\middle|\, \mathcal{F}_1\right] = S_1^1 \quad Q\text{-a.s.}$$

$$\Leftrightarrow \quad E_Q\left[S_0^1 Y_1/(1+r)\right] = S_0^1 \quad \text{and} \quad E_Q\left[S_0^1 Y_1 Y_2/(1+r)^2 \,\middle|\, \mathcal{F}_1\right] = S_0^1 Y_1/(1+r) \quad Q\text{-a.s.}$$

$$\Leftrightarrow \quad E_Q\left[Y_1\right] = 1 + r \quad \text{and} \quad E_Q\left[Y_2 \,|\, \mathcal{F}_1\right] = 1 + r \quad Q\text{-a.s.}$$

Since $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 takes three values, the latter is equivalent to

$$E_Q\left[Y_1\right] = 1 + r \quad \text{and} \quad E_Q\left[Y_2 \,|\, Y_1 = 1 + y_j\right] = 1 + r, \quad j = 1, 2, 3.$$

For the first equation we can compute

$$\begin{aligned} E_Q\left[Y_1\right] = 1 + r \quad &\Leftrightarrow \quad q_1 \times (1+d) + q_2 \times (1+m) + q_3 \times (1+u) = 1 + r \\ &\Leftrightarrow \quad q_1 \times d + q_2 \times m + q_3 \times u = r \\ &\Leftrightarrow \quad -0.01 q_1 + 0.01 q_2 + 0.03 q_3 = 0.01 \\ &\Leftrightarrow \quad -q_1 + q_2 + 3 q_3 = 1 \,. \end{aligned}$$

Since Q is a probability measure equivalent to P, the triplet (q_1, q_2, q_3) has to satisfy

$$\begin{cases} -q_1 + q_2 + 3q_3 & = & 1 \\ q_1 + q_2 + q_3 & = & 1 \\ q_1, q_2, q_3 \in (0,1) \end{cases}$$

Subtracting the second equation from the first yields

$$2q_3 - 2q_1 = 0 \quad \Leftrightarrow \quad q_1 = q_3 \,.$$

This in turn implies $q_2 = 1 - 2q_1$, and by the positivity constraint $0 < q_1 < 0.5$. In conclusion, $(q_1, q_2, q_3) \in (0,1)^3$ satisfies all the required conditions if and only if it is of the form $(\lambda, 1 - 2\lambda, \lambda)$, where $0 < \lambda < 0.5$.

For the second equation note that we have again

$$E_Q\left[Y_2 \,|\, Y_1 = 1 + y_j\right] = 1 + r \quad \Leftrightarrow \quad q_{j1} \times (1 + d) + q_{j2} \times (1 + m) + q_{j3} \times (1 + u) = 1 + r.$$

Using the first part, we may thus conclude that (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3})$, $j = 1$, 2, 3, describe a EMM for S^1 if and only if they are of the form $(\lambda, 1 - 2\lambda, \lambda)$, $(\mu_j, 1 - 2\mu_j, \mu_j)$, where $0 < \lambda, \mu_j < 0.5$, $j = 1$, 2, 3.

Note that while the condition for the martingale property is the same in each node, this condition is satisfied by many triplets, and we are allowed to choose a different triplet for each node. In other words, transition probabilities for Q need not be homogeneous across nodes, or equivalently put, we may choose λ, μ_1, μ_2, μ_3 all different.