Mathematical Foundations for Finance

Exercise Sheet 4

Please hand in your solutions by 12:00 on Wednesday, October 23 via the course homepage.

Exercise 4.1 Let (S^0, S^1) be the (discounted) binomial model with $T = 1$, $p \in (0, 1)$, and $u > 0 > d > -1$. Fix some $K > 0$, and define the functions $h_C, h_P : \mathbb{R} \to \mathbb{R}$ by

$$h_C(x) := (x - K)^+ := \max\{0, x - K\},$$
$$h_P(x) := (K - x)^+ := \max\{0, K - x\}.$$

The European options with payoff functions h_C and h_P are called the *European call option* and the *European put option*, respectively.

(a) Construct a self-financing strategy $\varphi^C \mathrel{\widehat{=}} (V_0^C, \vartheta^C)$ such that

$$V_1(\varphi^C) = h_C(S_1^1).$$

Write down explicitly the values of V_0^C and ϑ_1^C.

(b) Construct a self-financing strategy $\varphi^P \mathrel{\widehat{=}} (V_0^P, \vartheta^P)$ such that

$$V_1(\varphi^P) = h_P(S_1^1).$$

Write down explicitly the values of V_0^P and ϑ_1^P.

(c) Prove the *put–call parity* relation

$$V_0^P - V_0^C = K - S_0^1.$$

Exercise 4.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and Y a random variable normally distributed such that $Y \sim \mathcal{N}(0, 1)$.

(a) Fix a constant $\beta \in (0, \frac{1}{2})$, and consider the random variable

$$Z := \exp\left(-(\tfrac{1}{2} - \beta)Y - \frac{(\frac{1}{2} - \beta)^2}{2}\right).$$

Define the map $Q : \mathcal{F} \to \mathbb{R}$ by $Q[A] := E[Z\mathbb{1}_A]$. Prove that Q is a probability measure on $(\Omega, \mathcal{F})$, and that it is equivalent to P.

(b) Set

$$S_0^1 := e^{\beta} \qquad \text{and} \qquad S_1^1 := e^{Y}.$$

Prove that Q is an equivalent martingale measure for $S^1 = (S_0^1, S_1^1)$, with respect to the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

Hint: The statement $Q[A] = E[Z\mathbb{1}_A]$ *for all* $A \in \mathcal{F}$ *is equivalent to the statement* $E_Q[U] = E[ZU]$ *for all nonnegative random variables* U.

(c) Now consider the market (S^0, S^1), where $S^0 \equiv 1$ represents a bank account and S^1 is as in part (b). Fix some $K > 0$ and define the function $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) = (x - K)^+ := \max\{x - K, 0\}.$$

Compute $V_0^C := E_Q[C(S_1^1)]$ in terms of the cumulative distribution function of a standard normal random variable.