Mathematical Foundations for Finance

Exercise Sheet 4

Exercise 4.1 Let (S^0, S^1) be the (discounted) binomial model with $T = 1$, $p \in (0, 1)$, and $u > 0 > d > -1$. Fix some $K > 0$, and define the functions $h_C, h_P : \mathbb{R} \to \mathbb{R}$ by

$$h_C(x) := (x - K)^+ := \max\{0, x - K\},$$
$$h_P(x) := (K - x)^+ := \max\{0, K - x\}.$$

The European options with payoff functions h_C and h_P are called the *European call option* and the *European put option*, respectively.

(a) Construct a self-financing strategy $\varphi^C \widehat{=} (V_0^C, \vartheta^C)$ such that

$$V_1(\varphi^C) = h_C(S_1^1).$$

Write down explicitly the values of V_0^C and ϑ_1^C.

(b) Construct a self-financing strategy $\varphi^P \widehat{=} (V_0^P, \vartheta^P)$ such that

$$V_1(\varphi^P) = h_P(S_1^1).$$

Write down explicitly the values of V_0^P and ϑ_1^P.

(c) Prove the *put–call parity* relation

$$V_0^P - V_0^C = K - S_0^1.$$

Solution 4.1

(a) Consider a self-financing strategy $\varphi^C \widehat{=} (V_0^C, \vartheta^C)$. By definition,

$$V_1(\varphi^C) = V_0^C + \vartheta_1^C \Delta S_1^1.$$

Since (S^0, S^1) is the binomial model, we have that either $S_1^1 = (1 + u)S_0^1$ or $S_1^1 = (1 + d)S_0^1$. Also, since ϑ_1^C is $\mathcal{F}_0$-measurable, it is a constant (i.e. non-random). Thus, φ satisfies $V_1(\varphi^C) = h_C(S_1^1)$ if and only if

$$V_0^C + \vartheta_1^C u S_0^1 = h_C\Big((1 + u)S_0^1\Big),$$
$$V_0^C + \vartheta_1^C d S_0^1 = h_C\Big((1 + d)S_0^1\Big).$$

Subtracting the two equalities and rearranging gives

$$\vartheta_1^C = \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1}.$$

It remains to find V_0^C, which we can do by substituting the value of ϑ_1^C into either of the two previous equalities (we choose the first one) to get

$$\begin{aligned} V_0^C &= h_C\Big((1+u)S_0^1\Big) - \vartheta_1^C u S_0^1 \\ &= h_C\Big((1+u)S_0^1\Big) - \frac{h_C((1+u)S_0^1) - h_C((1+d)S_0^1)}{(u-d)S_0^1} u S_0^1 \\ &= \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big). \end{aligned}$$

Note. Since $\frac{u}{u-d} + \frac{-d}{u-d} = 1$ *and* $\frac{u}{u-d} \in (0,1)$*, we can also write* $V_0^C = E^*[h_C(S_1^1)]$*, where* E^* *denotes the expectation under the "risk-neutral" probability measure* P^* *given by*

$$P^*[S_1^1 = (1+d)S_0^1] = \frac{u}{u-d}, \qquad P^*[S_1^1 = (1+u)S_0^1] = 1 - \frac{u}{u-d} = \frac{-d}{u-d}.$$

(b) The same reasoning as in part (a) yields

$$\begin{aligned} \vartheta_1^P &= \frac{h_P((1+u)S_0^1) - h_P((1+d)S_0^1)}{(u-d)S_0^1}, \\ V_0^P &= \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big). \end{aligned}$$

Note. For the same risk-neutral probability measure P^* *as in part (a), we can write*

$$V_0^P = E^*[h_P(S_1^1)].$$

(c) First we compute, for $x \in \mathbb{R}$,

$$h_P(x) - h_C(x) = \max\{0, K-x\} - \max\{0, x-K\} = K - x.$$

Using this together with parts (a) and (b) yields

$$\begin{aligned} V_0^P - V_0^C &= \frac{u}{u-d} h_P\Big((1+d)S_0^1\Big) + \frac{-d}{u-d} h_P\Big((1+u)S_0^1\Big) \\ &\quad - \frac{u}{u-d} h_C\Big((1+d)S_0^1\Big) - \frac{-d}{u-d} h_C\Big((1+u)S_0^1\Big) \\ &= \frac{u}{u-d}\Big(K - (1+d)S_0^1\Big) + \frac{-d}{u-d}\Big(K - (1+u)S_0^1\Big) \\ &= K - S_0^1, \end{aligned}$$

as required.

Alternatively, we could use the expectation under the risk-neutral measure to get

$$V_0^P - V_0^C = E^*[h_P(S_1^1) - h_C(S_1^1)] = E^*[K - S_1^1] = K - E^*[S_1^1].$$

We then compute

$$\begin{aligned} E^*[S_1^1] &= (1+d)S_0^1 P^*[S_1^1 = (1+d)S_0^1] + (1+u)S_0^1 P^*[S_1^1 = (1+u)S_0^1] \\ &= (1+d)S_0^1 \frac{u}{u-d} + (1+u)S_0^1 \frac{-d}{u-d} \\ &= S_0^1, \end{aligned}$$

and hence

$$V_0^P - V_0^C = K - S_0^1,$$

as required.

Exercise 4.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and Y a random variable normally distributed such that $Y \sim \mathcal{N}(0, 1)$.

(a) Fix a constant $\beta \in (0, \frac{1}{2})$, and consider the random variable

$$Z := \exp\left(-(\tfrac{1}{2} - \beta)Y - \frac{(\frac{1}{2} - \beta)^2}{2} \right).$$

Define the map $Q : \mathcal{F} \to \mathbb{R}$ by $Q[A] := E[Z\mathbb{1}_A]$. Prove that Q is a probability measure on $(\Omega, \mathcal{F})$, and that it is equivalent to P.

(b) Set

$$S_0^1 := e^\beta \qquad \text{and} \qquad S_1^1 := e^Y.$$

Prove that Q is an equivalent martingale measure for $S^1 = (S_0^1, S_1^1)$, with respect to the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ given by $\mathcal{F}_0 := \{\varnothing, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

Hint: The statement $Q[A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{F}$ is equivalent to the statement $E_Q[U] = E[ZU]$ for all nonnegative random variables U.

(c) Now consider the market (S^0, S^1), where $S^0 \equiv 1$ represents a bank account and S^1 is as in part (b). Fix some $K > 0$ and define the function $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) = (x - K)^+ := \max\{x - K, 0\}.$$

Compute $V_0^C := E_Q[C(S_1^1)]$ in terms of the cumulative distribution function of a standard normal random variable.

Solution 4.2

(a) In order for Q to be a probability measure, we need to verify that

- $Q[A] \in [0,1]$ for all $A \in \mathcal{F}$;
- $Q[\varnothing] = 0$;
- $Q[\cup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} Q[A_n]$ for any disjoint family of sets $(A_n)_{n\in\mathbb{N}}$.

First, note that since Y is a standard normal random variable, Z is integrable (and nonnegative), and thus Q is a well-defined function with values in $[0,\infty)$. Moreover,

$$Q[\Omega] = E[Z\mathbb{1}_\Omega] = E[Z] = \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)\right] = 1,$$

since the moment-generating function of a standard normal random variable is $\phi(t) = E[e^{tY}] = \exp(\frac{t^2}{2})$. Also, we have $Q[\varnothing] = E[Z\mathbb{1}_\varnothing] = E[0] = 0$. Next, if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then $\mathbb{1}_{\bigcup_{n=1}^\infty A_n} = \sum_{n=1}^\infty \mathbb{1}_{A_n}$, and so

$$Q\left[\bigcup_{n=1}^{\infty} A_n\right] = E\left[Z\mathbb{1}_{\bigcup_{n=1}^\infty A_n}\right] = E\left[\sum_{n=1}^{\infty} Z\mathbb{1}_{A_n}\right].$$

Since Z is nonnegative, then $0 \leqslant \sum_{n=1}^{N} Z\mathbb{1}_{A_n} \uparrow \sum_{n=1}^{\infty} Z\mathbb{1}_{A_n}$ as $N \to \infty$, and thus the monotone convergence theorem implies that

$$E\left[\sum_{n=1}^{\infty} Z\mathbb{1}_{A_n}\right] = \lim_{N\to\infty} E\left[\sum_{n=1}^{N} Z\mathbb{1}_{A_n}\right] = \lim_{N\to\infty} \sum_{n=1}^{N} E[Z\mathbb{1}_{A_n}] = \sum_{n=1}^{\infty} E[Z\mathbb{1}_{A_n}].$$

Hence, we have

$$Q\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} Q[A_n].$$

We can conclude that Q is a probability measure on $(\Omega, \mathcal{F})$.

It remains to show that $Q \approx P$. To this end, let $A \in \mathcal{F}$ with $P[A] = 0$. Then $Z\mathbb{1}_A = 0$ P-a.s, and thus

$$Q[A] = E[Z\mathbb{1}_A] = E[0] = 0.$$

Hence, $Q \ll P$. Conversely, suppose that $A \in \mathcal{F}$ with $Q[A] = 0$. This means that $E[Z\mathbb{1}_A] = 0$. Since $Z\mathbb{1}_A$ is nonnegative, then $Z\mathbb{1}_A = 0$ P-a.s. Also, since $Z > 0$, then $Z\mathbb{1}_A = 0$ exactly when $\mathbb{1}_A = 0$, and so $\mathbb{1}_A = 0$ P-a.s, i.e. $P[A] = 0$. It follows that $P \ll Q$, and hence $Q \approx P$, as required.

(b) Since $Q \approx P$ by part (a), it remains to show that S^1 is a Q-martingale. It is immediate that S^1 is $\mathbb{F}$-adapted. Also, since $S_1^1 \geqslant 0$, we have $E[|S_1^1|] = E[S_1^1]$, and by the hint,

$$E_Q[S_1^1] = E[ZS_1^1] = E\left[\exp\left((\tfrac{1}{2}+\beta)Y - \frac{(\frac{1}{2}-\beta)^2}{2}\right)\right]$$
$$= \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left((\tfrac{1}{2}+\beta)Y\right)\right] = \exp\left(\frac{(\frac{1}{2}+\beta)^2-(\frac{1}{2}-\beta)^2}{2}\right)$$
$$= e^\beta < \infty,$$

using again that $E[e^{tY}] = \exp(\frac{t^2}{2})$. Thus, S^1 is Q-integrable. Finally, we note that since $\mathcal{F}_0$ is the trivial σ-field, then by Exercise 1.3(c),

$$E_Q[S_1^1 \mid \mathcal{F}_0] = E_Q[S_1^1] = e^\beta = S_0^1.$$

This completes the proof.

(c) We have that

$$V_0^C = E_Q[C(S_1^1)] = E_Q[(S_1^1-K)^+] = E[Z(S_1^1-K)^+]$$
$$= \exp\left(-\frac{(\frac{1}{2}-\beta)^2}{2}\right) E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)(e^Y-K)\mathbb{1}_{\{e^Y>K\}}\right].$$

Since the distribution of Y has density $f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})$, and also since $\{e^Y > K\} = \{Y > \log K\}$, we can write

$$E\left[\exp\left(-(\tfrac{1}{2}-\beta)Y\right)(e^Y-K)\mathbb{1}_{\{e^Y>K\}}\right] = \frac{1}{\sqrt{2\pi}}\int_{\log K}^\infty e^{-(\frac{1}{2}-\beta)y}(e^y-K)e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

The integrand can be rewritten as

$$e^{-(\frac{1}{2}-\beta)y}(e^y-K)e^{-\frac{y^2}{2}} = e^{(\frac{1}{2}+\beta)y-\frac{y^2}{2}} - Ke^{-(\frac{1}{2}-\beta)y-\frac{y^2}{2}}$$
$$= e^{-\frac{(y-(\beta+\frac{1}{2}))^2}{2}}e^{\frac{(\beta+\frac{1}{2})^2}{2}} - Ke^{-\frac{(y-(\beta-\frac{1}{2}))^2}{2}}e^{\frac{(\beta-\frac{1}{2})^2}{2}},$$

and so

$$\frac{1}{\sqrt{2\pi}}\int_{\log K}^\infty e^{-(\frac{1}{2}-\beta)y}(e^y-K)e^{-\frac{y^2}{2}}\,\mathrm{d}y = e^{\frac{(\beta+\frac{1}{2})^2}{2}}\int_{\log K}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-(\beta+\frac{1}{2}))^2}{2}}\,\mathrm{d}y$$
$$- Ke^{\frac{(\beta-\frac{1}{2})^2}{2}}\int_{\log K}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-(\beta-\frac{1}{2}))^2}{2}}\,\mathrm{d}y$$
$$= e^{\frac{(\beta+\frac{1}{2})^2}{2}}P[Y+\beta+\tfrac{1}{2} > \log K]$$
$$- Ke^{\frac{(\beta-\frac{1}{2})^2}{2}}P[Y+\beta-\tfrac{1}{2} > \log K],$$

since the above two integrands are the densities of the random variables $Y+\beta+\frac{1}{2}$ and $Y+\beta-\frac{1}{2}$, respectively. Letting $\Phi(x) := P[Y \leqslant x]$ denote the cumulative distribution function of a standard normal random variable, we can rewrite the above as

$$\frac{1}{\sqrt{2\pi}}\int_{\log K}^{\infty} e^{-(\frac{1}{2}-\beta)y}(e^y - K)e^{-\frac{y^2}{2}}\,\mathrm{d}y = e^{\frac{(\beta+\frac{1}{2})^2}{2}}\Phi(-\log K + \beta + \tfrac{1}{2}) - e^{\frac{(\beta-\frac{1}{2})^2}{2}}K\Phi(-\log K + \beta - \tfrac{1}{2}).$$

Remembering that V_0^C is the product of $\exp(-\frac{(\frac{1}{2}-r)^2}{2})$ and the above difference, we get

$$\begin{aligned} V_0^C &= e^{\frac{-(\frac{1}{2}-\beta)^2+(\beta+\frac{1}{2})^2}{2}}\Phi(-\log K + \beta + \tfrac{1}{2}) - K\Phi(-\log K + \beta - \tfrac{1}{2}) \\ &= e^{\beta}\Phi(-\log K + \beta + \tfrac{1}{2}) - K\Phi(-\log K + \beta - \tfrac{1}{2}). \end{aligned}$$

This completes the problem.