Mathematical Foundations for Finance

Exercise Sheet 5

Please hand in your solutions by 12:00 on Wednesday, October 30 via the course homepage.

Exercise 5.1 On a probability space $(\Omega, \mathcal{F}, P)$, consider a random variable X which is uniformly distributed on $(0,1)$. Let $Y = (Y_k)_{k=0}^2$ be the process given by

$$Y_0 = 0, \quad Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2 = X - \frac{1}{2} + \frac{B}{X^2}$$

for some random variable B independent of X and such that $P[B = 1] = P[B = -1] = 1/2$. Finally define the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0}^2$ by $\mathcal{F}_k := \sigma(Y_i,\ i \leq k)$.

(a) Prove that Y is not a martingale.

(b) Consider the sequence $(\tau_n)_{n\in\mathbb{N}}$ given by $\tau_n := \mathbb{1}_{\{X \geq 1/n\}} + 1$. Show that it is a sequence of stopping times increasing to 2 with $P[\tau_n = 2] \to 1$ as $n \to \infty$.

(c) Prove that Y is a local martingale by showing that $(\tau_n)_{n\in\mathbb{N}}$ can be chosen as localising sequence.

Exercise 5.2 Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T})$ be a filtered probability space and $S = (S_k)_{k=0,\ldots,T}$ a discounted price process. Show that the following are equivalent:

(a) S satisfies (NA).

(b) For each $k = 0, \ldots, T-1$, the one-period market (S_k, S_{k+1}) on $(\Omega, \mathcal{F}_{k+1}, P, (\mathcal{F}_k, \mathcal{F}_{k+1}))$ satisfies (NA).

Give an economic interpretation of this result.

Hint: Prove the contraposition of the direction "(b) $\Rightarrow$ (a)". Argue via induction on T.