Mathematical Foundations for Finance

Exercise Sheet 5

Exercise 5.1 On a probability space $(\Omega, \mathcal{F}, P)$, consider a random variable X which is uniformly distributed on $(0,1)$. Let $Y = (Y_k)_{k=0}^2$ be the process given by

$$Y_0 = 0, \quad Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2 = X - \frac{1}{2} + \frac{B}{X^2}$$

for some random variable B independent of X and such that $P[B = 1] = P[B = -1] = 1/2$. Finally define the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0}^2$ by $\mathcal{F}_k := \sigma(Y_i,\ i \leq k)$.

(a) Prove that Y is not a martingale.

(b) Consider the sequence $(\tau_n)_{n\in\mathbb{N}}$ given by $\tau_n := \mathbb{1}_{\{X \geq 1/n\}} + 1$. Show that it is a sequence of stopping times increasing to 2 with $P[\tau_n = 2] \to 1$ as $n \to \infty$.

(c) Prove that Y is a local martingale by showing that $(\tau_n)_{n\in\mathbb{N}}$ can be chosen as localising sequence.

Solution 5.1

(a) First note that

$$E\left[\left|\frac{1}{X^2}\right|\right] = E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2}\, dx = \infty.$$

By the triangle inequality, we get $|Y_2| \geq \left|\frac{1}{X^2}\right| - \left|X - \frac{1}{2}\right|$ and hence

$$E[|Y_2|] \geq E\left[\left|\frac{1}{X^2}\right|\right] - E\left[\left|X - \frac{1}{2}\right|\right] = \infty,$$

which shows that Y_2 is not integrable and thus that Y is not a martingale.

(b) We first check that τ_n is a stopping time for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and note that since $\tau_n \geq 1$, we already have that $\{\tau_n \leq 0\} = \emptyset \in \mathcal{F}_0$. Next note that X is $\mathcal{F}_1$-measurable, so that we also get $\{X < 1/n\} \in \mathcal{F}_1$. Hence we can write

$$\{\tau_n \leq 1\} = \{X < 1/n\} \in \mathcal{F}_1 \qquad \text{and} \qquad \{\tau_n \leq 2\} = \Omega \in \mathcal{F}_2,$$

and thus conclude that τ_n is a stopping time.

Since $\mathbb{1}_{\{X\geq 1/n\}} \leq \mathbb{1}_{\{X\geq 1/(n+1)\}}$, we already see that $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence. Moreover, noting that for each $\omega \in \Omega$ there exists an $n \in \mathbb{N}$ such that $X(\omega) \geq 1/n$, we also have that $\lim_{n\to\infty} \tau_n = 2$ P-a.s., and we can thus conclude that $(\tau_n)_{n\in\mathbb{N}}$ is increasing to 2. Finally, observe that $P[\tau_n = 2] = P[X \geq 1/n] = 1 - 1/n \to 1$ for $n \to \infty$.

(c) In order to prove that Y is a local martingale, we will show that $Y^{\tau_n} = (Y_{k\wedge\tau_n})_{k=0}^2$ is a martingale. Fix $n \in \mathbb{N}$ and note that $Y_0^{\tau_n} = Y_0 = 0$,

$$Y_1^{\tau_n} = Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2^{\tau_n} = Y_1 \mathbb{1}_{\{X<1/n\}} + Y_2 \mathbb{1}_{\{X\geq 1/n\}} = X - \frac{1}{2} + \frac{B}{X^2}\mathbb{1}_{\{X\geq 1/n\}}.$$

One can easily see that the process Y^{τ_n} is adapted and $Y_0^{\tau_n}$ and $Y_1^{\tau_n}$ are integrable, since X is bounded. Moreover, we can compute

$$E[|Y_2^{\tau_n}|] \leq E\left[\left|X - \frac{1}{2}\right|\right] + E\left[\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}}\right] \leq \frac{1}{4} + n^2 < \infty,$$

and thus conclude that $Y_k^{\tau_n}$ is integrable for all $k \in \{0, 1, 2\}$.

For the martingale condition, first note that $E[X] = \int_0^1 x\, dx = 1/2$ and hence

$$E[Y_1^{\tau_n}|\mathcal{F}_0] = E[X] - 1/2 = 0 = Y_0^{\tau_n}.$$

Moreover, using again that X is $\mathcal{F}_1$-measurable, we also have that

$$E[Y_2^{\tau_n}|\mathcal{F}_1] = E\left[X - \frac{1}{2} + \frac{B}{X^2}\mathbb{1}_{\{X\geq 1/n\}}\middle|\mathcal{F}_1\right] = X - \frac{1}{2} + E[B|\mathcal{F}_1]\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}} \quad P\text{-a.s.}$$

Noting that since B is independent of X, it is also independent of $\mathcal{F}_1$, we can deduce that

$$E[Y_2^{\tau_n}|\mathcal{F}_1] = X - \frac{1}{2} + E[B]\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}} = X - \frac{1}{2} = Y_1^{\tau_n} \quad P\text{-a.s.}$$

We can thus conclude that Y is a local martingale with localising sequence $(\tau_n)_{n\in\mathbb{N}}$.

Exercise 5.2 Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T})$ be a filtered probability space and $S = (S_k)_{k=0,\dots,T}$ a discounted price process. Show that the following are equivalent:

(a) S satisfies (NA).

(b) For each $k = 0, \dots, T-1$, the one-period market (S_k, S_{k+1}) on $(\Omega, \mathcal{F}_{k+1}, P, (\mathcal{F}_k, \mathcal{F}_{k+1}))$ satisfies (NA).

Give an economic interpretation of this result.

Hint: Prove the contraposition of the direction "(b) $\Rightarrow$ (a)". Argue via induction on T.

Solution 5.2

(a) "(a) $\Rightarrow$ (b)": We prove the contraposition. Let ϑ_k be an $\mathcal{F}_{k-1}$-measurable random variable such that $\vartheta_k \Delta S_k \geq 0$ P-a.s. and $P[\vartheta_k \Delta S_k > 0] > 0$. Extending ϑ_k to a predictable process $\widehat{\vartheta}$ via

$$\widehat{\vartheta}_j := \begin{cases} \vartheta_k, & \text{if } j = k, \\ 0, & \text{for } j \in \{1, \ldots, T\} \backslash \{k\}, \end{cases}$$

we obtain that $G_T(\widehat{\vartheta}) = \vartheta_k \Delta S_k \geq 0$ P-a.s. and hence also $P\left[G_T(\widehat{\vartheta}) > 0\right] > 0$. This means that arbitrage in the "small" market yields arbitrage in the "big" market.

"(b) $\Rightarrow$ (a)": Again, we prove the contraposition. Let ϑ be an arbitrage opportunity, i.e.

$$(\vartheta \cdot S)_T \geq 0 \quad P\text{-a.s.} \quad \text{and} \quad P[(\vartheta \cdot S)_T > 0] > 0\,.$$

We claim that there exist $k \in \{1, \ldots, T\}$ and $A \in \mathcal{F}_{k-1}$ such that $P[A] > 0$, $\mathbb{1}_A \vartheta_k \Delta S_k \geq 0$ P-a.s. and $P[\mathbb{1}_A \vartheta_k \Delta S_k > 0] > 0$.

Proof: We prove the statement by induction on T. For $T = 1$, the situation is trivially satisfied. Suppose the assertion holds for $T - 1$. We distinguish three possibilities:

(i) $P[(\vartheta \cdot S)_{T-1} < 0] > 0$,

(ii) $P[(\vartheta \cdot S)_{T-1} = 0] = 1$ and

(iii) $(\vartheta \cdot S)_{T-1} \geq 0$ P-a.s. and $P[(\vartheta \cdot S)_{T-1} > 0] > 0$.

In case (i), we define $A := \{(\vartheta \cdot S)_{T-1} < 0\} \in \mathcal{F}_{T-1}$ and the strategy

$$\vartheta_k^A := \begin{cases} \mathbb{1}_A \vartheta_T, & k = T, \\ 0, & \text{for } k \in \{1, \ldots, T-1\}\,. \end{cases}$$

Because $(\vartheta^A \cdot S)_T = \mathbb{1}_A \vartheta_T \Delta S_T = \mathbb{1}_A \left((\vartheta \cdot S)_T - (\vartheta \cdot S)_{T-1}\right)$, this strategy has $(\vartheta^A \cdot S)_T \geq 0$ P-a.s. and $P\left[(\vartheta^A \cdot S)_T > 0\right] > 0$.

In case (ii), the choice $A = \Omega$ obviously yields an arbitrage opportunity in the one-period market (S_{T-1}, S_T) on $(\Omega, \mathcal{F}_T, P, (\mathcal{F}_{T-1}, \mathcal{F}_T))$.

In the remaining case (iii), we apply the inductive hypothesis.

It remains to give the interpretation. The result tells us that in order for a financial market to be free of arbitrage, it is necessary and sufficient that the local models (S_k, S_{k+1}) are arbitrage-free. Thus, the notion of (NA), which is a priori globally defined, turns out to be of local nature. On the other hand, if one knows the fundamental theorem of asset pricing, then one realizes that this local notion translates into nothing else than the following:

In order to check whether an adapted and integrable process S is a martingale, it suffices to check whether

$E_Q\left[S_{k+1} \,\middle|\, \mathcal{F}_k\right] = S_k \quad \forall k$ (local behaviour) instead of $E_Q\left[S_T \,\middle|\, \mathcal{F}_k\right] = S_k \quad \forall k$ (global behaviour).

Remark: The equivalence in this exercise no longer holds for models with transaction costs.