Mathematical Foundations for Finance

Exercise Sheet 6

Please hand in your solutions by 12:00 on Wednesday, November 6 via the course homepage.

Exercise 6.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \leq k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k = 0, 1, \dots, T : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$. Let $X = (X_k)_{k \in \mathbb{N}_0}$ be an adapted and integrable process.

(a) Find the *Doob decomposition* of X. In other words, prove that there exist a martingale $M = (M_k)_{k \in \mathbb{N}_0}$ and an integrable and predictable process $A = (A_k)_{k \in \mathbb{N}_0}$ that are both null at zero, and such that

$$X = X_0 + M + A \ P\text{-a.s.}$$

Hint: You may define $M_k := \sum_{j=1}^{k} (X_j - E[X_j \mid \mathcal{F}_{j-1}])$, for $k \in \mathbb{N}$.

(b) Prove that M and A are unique up to P-a.s. equality.

Exercise 6.3 Let $W = (W_t)_{t \geq 0}$ and $W' = (W'_t)_{t \geq 0}$ be two *independent* Brownian motions (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$. Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, for $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha W + \sqrt{1-\alpha^2} W'$ is a BM for any $\alpha \in [0, 1]$.

(d) Show that the independence of W and W' in (c) cannot be omitted, i.e., if W and W' are *not* independent, then W^3 need not be a BM. Give an example.