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Mathematical Foundations for Finance

Exercise Sheet 6

Exercise 6.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \leq k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k = 0, 1, \dots, T : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Solution 6.1

(a) Fix a $k \in \{0, 1, \dots, T\}$. For any $j \in \{0, 1, \dots, k\}$, X_j is $\mathcal{F}_j$-measurable because X is adapted, which means that $\{X_j \in B\} = \{\omega \in \Omega : X_j(\omega) \in B\} \in \mathcal{F}_j \subset \mathcal{F}_k$. Moreover, by definition of $\tau_{X,B}$, we have

$$\{\tau_{X,B} \leq k\} = \Big\{X_j \in B \text{ for some } j \in \{0, 1, \dots, k\}\Big\} = \bigcup_{j=0}^{k} \{X_j \in B\} \in \mathcal{F}_k$$

because $\mathcal{F}_k$ as a σ-algebra is closed under countable (and therefore also finite) unions. $\tau_{X,B}$ can, however, attain the value of $+\infty$ and thus does not satisfy the definition of a stopping time. However, since $\tau_{X,B} \wedge T$ can only attain values in $\{0, 1, \dots, T\}$ and

$$\{\tau_{X,B} \wedge T \leq k\} = \begin{cases} \{\tau_{X,B} \leq k\} & \text{for } k < T \\ \Omega & \text{for } k = T, \end{cases}$$

we have that $\tau_{X,B} \wedge T$ indeed is a stopping time.

(b) Given a stopping time τ, we define

$$X_k := \mathbb{1}_{\{\tau > k\}}$$

for $k = 0, 1, \dots, T$ and set $X := (X_k)_{k=0,1,\dots,T}$. Since τ is a stopping time, $\{\tau \leq k\}$ (and therefore $\{\tau > k\} = \{\tau \leq k\}^c$) is in $\mathcal{F}_k$ for every $k = 0, 1, \dots, T$. This implies that X is adapted. Moreover, $X_k(\omega) = 1$ for $\tau(\omega) > k$ and $X_k(\omega) = 0$ for $\tau(\omega) \leq k$ so that

$$\tau(\omega) = \inf\Big\{k = 0, \dots, T : X_k(\omega) \in \{0\}\Big\}.$$

Therefrom we clearly see that $\tau = \tau_{X,\{0\}}$ for an adapted $X = (X_k)_{k=0,1,\dots,T}$ defined by $X_k = \mathbb{1}_{\{\tau > k\}}$. Note that $\tau \leq T$, so $X_T = 0$ and hence $\tau_{X,\{0\}} \leq T$.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$. Let $X = (X_k)_{k \in \mathbb{N}_0}$ be an adapted and integrable process.

(a) Find the *Doob decomposition* of X. In other words, prove that there exist a martingale $M = (M_k)_{k \in \mathbb{N}_0}$ and an integrable and predictable process $A = (A_k)_{k \in \mathbb{N}_0}$ that are both null at zero, and such that

$$X = X_0 + M + A \ P\text{-a.s.}$$

Hint: You may define $M_k := \sum_{j=1}^{k} (X_j - E[X_j \mid \mathcal{F}_{j-1}])$, *for* $k \in \mathbb{N}$.

(b) Prove that M and A are unique up to P-a.s. equality.

Solution 6.2 To simplify notation, we omit "P-a.s." from all equalities below.

(a) For each $k \in \mathbb{N}_0$, take

$$M_k := \sum_{j=1}^{k} (X_j - E[X_j \mid \mathcal{F}_{j-1}]).$$

It is immediate that M is adapted, integrable, and null at zero. Then, for $k \in \mathbb{N}$, we have

$$\begin{aligned} E[M_k - M_{k-1} \mid \mathcal{F}_{k-1}] &= E\Big[X_k - E[X_k \mid \mathcal{F}_{k-1}] \,\Big|\, \mathcal{F}_{k-1}\Big] \\ &= E[X_k \mid \mathcal{F}_{k-1}] - E[X_k \mid \mathcal{F}_{k-1}] \\ &= 0. \end{aligned}$$

Hence, M is a martingale. Next, for each $k \in \mathbb{N}_0$, we set

$$\begin{aligned} A_k := X_k - X_0 - M_k &= X_k - X_0 - \sum_{j=1}^{k} (X_j - E[X_j \mid \mathcal{F}_{j-1}]) \\ &= \sum_{j=1}^{k} (E[X_j \mid \mathcal{F}_{j-1}] - X_{j-1}). \end{aligned}$$

Then A is predictable with $A_0 = 0$, and of course $X = X_0 + M + A$, as required.

(b) Suppose the processes $M^{(1)}, A^{(1)}$ and $M^{(2)}, A^{(2)}$ both satisfy the conditions of the problem. Subtracting the equalities

$$X - X_0 = M^{(1)} + A^{(1)},$$
$$X - X_0 = M^{(2)} + A^{(2)}$$

gives

$$M^{(1)} - M^{(2)} = A^{(2)} - A^{(1)}.$$

For notational convenience, we set $Y := M^{(1)} - M^{(2)} = A^{(2)} - A^{(1)}$. Since $Y = A^{(2)} - A^{(1)}$, then Y is predictable, and hence for all $k \in \mathbb{N}$,

$$Y_k = E[Y_k \mid \mathcal{F}_{k-1}].$$

But since the difference of two martingales is a martingale, Y is a martingale, and hence the above can be rewritten as

$$Y_k = Y_{k-1} \quad \forall k \in \mathbb{N}.$$

Since $Y_0 = 0$, this implies that $Y_k = 0$ for all $k \in \mathbb{N}_0$, and hence

$$M^{(1)} = M^{(2)} \text{ and } A^{(1)} = A^{(2)}.$$

This completes the proof.

Exercise 6.3 Let $W = (W_t)_{t\geq 0}$ and $W' = (W'_t)_{t\geq 0}$ be two *independent* Brownian motions (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$. Show that

(a) $W^1 := -W$ is a BM.

(b) $W^2_t := W_{T+t} - W_T$, for $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha W + \sqrt{1-\alpha^2} W'$ is a BM for any $\alpha \in [0, 1]$.

(d) Show that the independence of W and W' in (c) cannot be omitted, i.e., if W and W' are *not* independent, then W^3 need not be a BM. Give an example.

Solution 6.3 We first recall the definition of a Brownian motion in order to know what needs to be checked. A *Brownian motion* with respect to P is a real-valued stochastic process $W = (W_t)_{t\geq 0}$ such that

(BM0) $W_0 = 0$ P-a.s.

(BM1) For any $n \in \mathbb{N}$ and any times $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increments $W_{t_i} - W_{t_{i-1}}$ are independent and normally distributed with variance $t_i - t_{i-1}$ under P, i.e.

$$W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}) \text{ for } i = 1, \ldots, n.$$

(BM2) W has P-a.s. continuous trajectories.

(a) We check (BM0), (BM1) and (BM2) separately.

(BM0) This is clear since $W_0^1 = -W_0 = 0$ P-a.s.

(BM1) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have, for $i = 1, \dots, n$, that

$$W_{t_i}^1 - W_{t_{i-1}}^1 = -(W_{t_i} - W_{t_{i-1}}),$$

which are independent under P. Since $X \sim \mathcal{N}(0, \sigma^2)$ if and only if $-X \sim \mathcal{N}(0, \sigma^2)$, we also conclude that $W_{t_i}^1 - W_{t_{i-1}}^1 \sim \mathcal{N}(0, t_i - t_{i-1})$.

(BM2) This is trivial, since $W^1 = -W$. The sign does not alter continuity.

(b) We check (BM0), (BM1) and (BM2) separately.

(BM0) We obviously have $W_0^2 = W_T - W_T = 0$ P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have for $i = 1, \dots, n$ that

$$W_{t_i}^2 - W_{t_{i-1}}^2 = W_{T+t_i} - W_T - (W_{T+t_{i-1}} - W_T) = W_{T+t_i} - W_{T+t_{i-1}}.$$

Denoting $t_i' = T + t_i$, we see from the definition (BM1′) that the increments of W^2 are independent under P, and since $t_i' - t_{i-1}' = t_i - t_{i-1}$, we also conclude that for all $i = 1, \dots, n$, we have

$$W_{t_i}^2 - W_{t_{i-1}}^2 \sim \mathcal{N}(0, t_i - t_{i-1}).$$

(BM2) This is again easy, since W^2 is simply W shifted in time by T minus a random variable which does not depend on t.

(c) We check (BM0), (BM1) and (BM2) separately.

(BM0) $W_0^3 = \alpha W_0 + \sqrt{1 - \alpha^2} W_0' = 0$ P-a.s., since both W_0 and W_0' are equal to 0 P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have, for $i = 1, \dots, n$, that

$$W_{t_i}^3 - W_{t_{i-1}}^3 = \alpha \left(W_{t_i} - W_{t_{i-1}} \right) + \sqrt{1 - \alpha^2} \left(W_{t_i}' - W_{t_{i-1}}' \right).$$

Since W and W' are independent under P, we conclude that the right-hand side is an independent family of random variables. Since W and W' are BMs, we additionally have that

$$W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}),\ i = 1, \dots, n,$$
$$W_{t_i}' - W_{t_{i-1}}' \sim \mathcal{N}(0, t_i - t_{i-1}),\ i = 1, \dots, n.$$

Recall the general fact that if $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \eta^2)$ are independent, then we have for any linear combination $s_1 X + s_2 Y$ that

$$s_1 X + s_2 Y \sim \mathcal{N}(0, s_1^2 \sigma^2 + s_2^2 \eta^2).$$

Using this, we conclude that

$$\alpha\Big(W_{t_i} - W_{t_{i-1}}\Big) + \sqrt{1-\alpha^2}\Big(W'_{t_i} - W'_{t_{i-1}}\Big) \sim \mathcal{N}(0, t_i - t_{i-1})$$

since

$$\alpha^2(t_i - t_{i-1}) + (1-\alpha^2)(t_i - t_{i-1}) = t_i - t_{i-1}.$$

(BM2) This is evident, since W^3 is a linear combination of two processes whose paths are P-a.s. continuous.

(d) Two possible choices are $W = \pm W'$. In this case, we have

$$W^3 = \left(\alpha \pm \sqrt{1-\alpha^2}\right) W,$$

which is not a Brownian motion because $W_1^3 \sim \mathcal{N}\left(0, (\alpha \pm \sqrt{1-\alpha^2})^2\right)$ and $\left(\alpha \pm \sqrt{1-\alpha^2}\right)^2 \neq 1$ in general.