Mathematical Foundations for Finance

Exercise Sheet 7

Please hand in your solutions by 12:00 on Wednesday, November 13 via the course homepage.

Exercise 7.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g\colon \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\Big(M_t + g(t)\Big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

Exercise 7.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales, but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: You may use the results from the Exercises 7.1(b) and 7.1(a), respectively.

(b) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Exercise 7.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

For any constants a, $b \in \mathbb{R}$ such that $a < 0 < b$, consider the function $\tau : \Omega \to [0, \infty]$ given by

$$\tau := \inf\{t \geqslant 0 : W_t \notin [a, b]\}.$$

(a) Show that τ is a stopping time.
Hint: You may use the right-continuity of the filtration $\mathbb{F}$.

(b) Prove that $E[W_\tau] = 0$.
Hint: You may apply the dominated convergence theorem.

(c) Compute $P[W_\tau = a]$.
Hint: You may use the result from (b).