Mathematical Foundations for Finance

Exercise Sheet 7

Exercise 7.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\Big(M_t + g(t)\Big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

Solution 7.1

(a) First recall that W is a $(P, \mathbb{F})$-martingale. Adaptedness is clear since f is continuous. Indeed, recall that any real-valued convex function is continuous on the interior of its domain. Integrability is assumed. Then by Jensen's inequality for conditional expectations, we can compute

$$E\left[f(W_t) \,|\, \mathcal{F}_s\right] \geq f\Big(E\left[W_t \,|\, \mathcal{F}_s\right]\Big) = f(W_s) \ P\text{-a.s.}$$

for all $t \geq s$, and thus conclude that $f(W)$ is a $(P, \mathbb{F})$-submartingale.

(b) For any measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, we have that, for each $t \geq 0$, $M_t + g(t)$ is $\mathcal{F}_t$-measurable and

$$E\left[|M_t + g(t)|\right] \leq E\left[|M_t|\right] + E\left[|g(t)|\right] = E\left[|M_t|\right] + |g(t)| < \infty.$$

Hence $\Big(M_t + g(t)\Big)_{t\geq 0}$ is adapted and integrable. We can then compute

$$E\left[M_t + g(t) \,|\, \mathcal{F}_s\right] = E\left[M_t \,|\, \mathcal{F}_s\right] + g(t) = M_s + g(s) + g(t) - g(s) \ P\text{-a.s.}$$

for all $t \geq s$. As a result, $\Big(M_t + g(t)\Big)_{t\geq 0}$ has the $(P, \mathbb{F})$-supermartingale property, i.e.

$$E\left[M_t + g(t) \,|\, \mathcal{F}_s\right] \leq M_s + g(s) \ P\text{-a.s.}$$

for all $t > s$, if and only if g is decreasing. Analogously, $\Big(M_t + g(t)\Big)_{t\geq 0}$ has the $(P, \mathbb{F})$-submartingale property, i.e.

$$E\left[M_t + g(t)\,|\,\mathcal{F}_s\right] \geq M_s + g(s)\ P\text{-a.s.}$$

for all $t > s$, if and only if g is increasing.

Exercise 7.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales, but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: You may use the results from the Exercises 7.1(b) and 7.1(a), respectively.

(b) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Solution 7.2

(a) (i) Using the notation of Exercise 7.1(b), we notice that $W_t^2 = W_t^2 - t + g(t)$, where $g(t) := t$, for $t \geq 0$. By Proposition IV.2.3 in the lecture notes, we know that $\Big(W_t^2 - t\Big)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, using that g is increasing, we can conclude that W^2 is a $(P, \mathbb{F})$-submartingale. In order to show that W^2 is not a martingale, we can use the martingale property of $\Big(W_t^2 - t\Big)_{t\geq 0}$ to compute

$$E\left[W_t^2 \,\middle|\, \mathcal{F}_s\right] = E\left[W_t^2 - t \,\middle|\, \mathcal{F}_s\right] + t = W_s^2 - s + t > W_s^2\ P\text{-a.s.},$$

showing that W^2 is not a $(P, \mathbb{F})$-martingale.

(ii) Adaptedness is clear since the transformation $x \mapsto e^{\alpha x}$ is continuous, and since we know that $W_t \stackrel{d}{=} W_t - W_0$ is $\mathcal{N}(0, t)$-distributed, the random variable $e^{\alpha W_t}$ is integrable for any $t \geq 0$. Noting that $x \mapsto e^{\alpha x}$ is also a convex function, we can then apply Exercise 7.1(a) to conclude that $e^{\alpha W}$ is a $(P, \mathbb{F})$-submartingale. Next, Proposition IV.2.3 in the lecture notes gives us that $\Big(e^{\alpha W_t - \frac{1}{2}\alpha^2 t}\Big)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, we can compute

$$E\left[e^{\alpha W_t} \,\middle|\, \mathcal{F}_s\right] = E\left[e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \,\middle|\, \mathcal{F}_s\right] e^{\frac{1}{2}\alpha^2 t} = e^{\alpha W_s} e^{\frac{1}{2}\alpha^2 (t-s)} > e^{\alpha W_s}\ P\text{-a.s.},$$

showing that $e^{\alpha W}$ is not a $(P, \mathbb{F})$-martingale.

(b) Let $(X_t)_{t \geq 0}$ be a $(P, \mathbb{F})$-local martingale null at 0 and uniformly bounded from below by $-a \leq 0$ and denote by $(\tau_n)_{n \in \mathbb{N}}$ a localizing sequence. Since $\lim_{n\to\infty} \tau_n = \infty$ P-a.s., we have

$$\lim_{n\to\infty} X_{t\wedge\tau_n} = X_t \ P\text{-a.s.}$$

Moreover, since $(X_t)_{t \geq 0}$ is uniformly bounded from below by $-a$, we have that $X_{t\wedge\tau_n} \geq -a$ and thus $0 \leq |X_{t\wedge\tau_n}| \leq X_{t\wedge\tau_n} + 2a$ for all $n \in \mathbb{N}$. By Fatou's lemma, we can then compute

$$\begin{aligned} E\left[|X_t|\right] = E\left[\lim_{n\to\infty} |X_{t\wedge\tau_n}|\right] &\leq \liminf_{n\to\infty} E\left[|X_{t\wedge\tau_n}|\right] \\ &\leq \liminf_{n\to\infty} E\left[X_{t\wedge\tau_n}\right] + 2a = 2a < \infty, \end{aligned}$$

where the last equality uses the martingale property of X^{τ_n} and the fact that it is null at 0. We have thus proved integrability. Since adaptedness is clear by the definition of a local martingale, it only remains to show the $(P, \mathbb{F})$-supermartingale property. Using again that $X_{t\wedge\tau_n} \geq -a$ for all $n \in \mathbb{N}$, we can apply Fatou's lemma to obtain for $t > s$

$$E\left[X_t \,|\, \mathcal{F}_s\right] = E\left[\lim_{n\to\infty} X_{t\wedge\tau_n} \,\middle|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} E\left[X_{t\wedge\tau_n} \,|\, \mathcal{F}_s\right] = X_s,$$

as desired.

Exercise 7.3 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions. For any constants a, $b \in \mathbb{R}$ such that $a < 0 < b$, consider the function $\tau : \Omega \to [0, \infty]$ given by

$$\tau := \inf\{t \geqslant 0 : W_t \notin [a, b]\}.$$

(a) Show that τ is a stopping time.
Hint: You may use the right-continuity of the filtration $\mathbb{F}$.

(b) Prove that $E[W_\tau] = 0$.
Hint: You may apply the dominated convergence theorem.

(c) Compute $P[W_\tau = a]$.
Hint: You may use the result from (b).

Solution 7.3

(a) Fix $s > 0$. We have

$$\{\tau < s\} = \bigcup_{r \in [0,s)} \{W_r \notin [a, b]\} = \bigcup_{r \in [0,s)\cap\mathbb{Q}} \{W_r \notin [a, b]\},$$

where in the last step we have used that W has continuous paths and the set $[a,b]$ is closed. Since W is adapted, then $\{W_r \in [a,b]\} \in \mathcal{F}_r \subseteq \mathcal{F}_s$ for all $r \in [0,s)$, and since $[0,s) \cap \mathbb{Q}$ is countable, we thus have that

$$\{\tau < s\} \in \mathcal{F}_s.$$

This implies that for all $t \geqslant 0$ and $N \in \mathbb{N}$,

$$\{\tau \leqslant t\} = \bigcap_{n=N}^{\infty} \{\tau < t + \tfrac{1}{n}\} \in \mathcal{F}_{t+\frac{1}{N}},$$

so that

$$\{\tau \leqslant t\} \in \bigcap_{N=1}^{\infty} \mathcal{F}_{t+\frac{1}{N}} = \mathcal{F}_{t+}.$$

Finally, since we have assumed that $\mathbb{F}$ satisfies the usual conditions (and in particular is right-continuous), then $\mathcal{F}_t = \mathcal{F}_{t+}$, and hence

$$\{\tau \leqslant t\} = \mathcal{F}_t.$$

This completes the proof.

(b) Fix $n \in \mathbb{N}$. We have that $\tau \wedge n$ is a bounded stopping time, and thus by the stopping theorem (Theorem IV.2.2 in the lecture notes),

$$E[W_{\tau\wedge n}] = E[W_0] = 0.$$

By the law of the iterated logarithm (Proposition IV.1.2 in the lecture notes), $\tau < \infty$ P-a.s., and hence

$$\lim_{n\to\infty} (\tau \wedge n) = \tau \ P\text{-a.s.},$$

so that $W_{\tau\wedge n} \to W_\tau$ P-a.s. as $n \to \infty$. Since $W_{\tau\wedge n} \in [a,b]$ by the definition of τ (and since $W_0 = 0$), we may apply the dominated convergence theorem to get

$$E[W_\tau] = \lim_{n\to\infty} E[W_{\tau\wedge n}] = 0,$$

as required.

(c) Since $\tau < \infty$ P-a.s., W_τ is either a or b (because W has continuous paths and starts at zero). It follows that

$$E[W_\tau] = aP[W_\tau = a] + bP[W_\tau = b],$$

and that

$$P[W_\tau = b] = 1 - P[W_\tau = a].$$

Substituting the latter equality into the former gives

$$E[W_\tau] = (a-b)P[W_\tau = a] + b.$$

Using part (b) and rearranging, we get

$$P[W_\tau = a] = \frac{b}{b-a}.$$

Note that from this we also immediately get

$$P[W_\tau = b] = \frac{-a}{b-a}.$$