Mathematical Foundations for Finance

Exercise Sheet 8

Please hand in your solutions by 12:00 on Wednesday, November 20 via the course homepage.

Exercise 8.1 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. On this space, let M be a local martingale null at 0 that satisfies $\sup_{0\leq t\leq T}|M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: You may use the dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Prove that

$$E\Big[[M]_t - [M]_s \,\Big|\, \mathcal{F}_s\Big] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s] \; P\text{-a.s., for } 0 \leq s \leq t \leq T.$$

Hint: You may use that $\mathrm{Var}[X \,|\, \mathcal{G}] = E\Big[(X - E\,[X\,|\mathcal{G}])^2 \,\Big|\, \mathcal{G}\Big]$.

Exercise 8.2 Let $(\Omega, \mathcal{F}, P)$ a probability space. We consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of square-integrable and independent random variables and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k)$ for all $k \in \mathbb{N}$. We assume that $(Y_k)_{k\in\mathbb{N}}$ are identically distributed, with $\mu := E\,[Y_k] \in \mathbb{R}$ and $\sigma^2 := \mathrm{Var}[Y_k] > 0$, for $k \in \mathbb{N}$. Define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by

$$X_n = \sum_{k=1}^{n} Y_k, \text{ for } n \in \mathbb{N}_0.$$

Note that X is adapted to $\mathbb{F}$ and integrable.

(a) Derive the Doob decomposition of X. In other words, find the martingale $M = (M_n)_{n\in\mathbb{N}_0}$ and the predictable and integrable process $A = (A_n)_{n\in\mathbb{N}_0}$ that are both null at zero and such that $X = M + A$ P-a.s. Deduce that M and A are square-integrable.
Hint: see Exercise 6.2(a).

(b) Find the optional quadratic variation $[M] = ([M]_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.
Hint: You may use Theorem 5.1.1 in the lecture notes, and in particular the condition $\Delta[M] = (\Delta M)^2$.

(c) Explicitly derive the predictable quadratic variation $\langle M\rangle = (\langle M\rangle_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.

Exercise 8.3

This exercise proves the frequently used fact that a continuous local martingale of finite variation is identically constant (and hence vanishes if it is null at 0).

For $p > 0$, the *(functional) p-variation* of a function $g : [0, \infty) \to \mathbb{R}$ is the function defined by

$$V^p(g) : [0, \infty) \to [0, \infty], \quad V^p_T(g) := \sup_\Pi V^p_T(g, \Pi) := \sup_\Pi \sum_{t_i \in \Pi} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p ,$$

where the supremum is taken over all partitions Π of $[0, \infty)$, i.e., over all sets $\Pi \subseteq [0, \infty)$ with $0 \in \Pi$ and $\Pi \cap [0, t]$ finite for all $t \geq 0$. A function g has finite *(functional) p-variation* if $V^p_T(g) < \infty$ for all $T \geq 0$, and *finite (functional) variation* if it has finite (functional) 1-variation. For $p = 2$, we also say (functional) "quadratic variation" instead of "2-variation". We say that g has zero p-variation along a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions if $\lim_{n\to\infty} V^p_T(g, \Pi_n) = 0$ for all $T \geq 0$. For $\Pi := (t_i)_{i\in\mathbb{N}}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{N}$, we also define $|\Pi| := \sup\{t_{i+1} - t_i \mid t_i, t_{i+1} \in \Pi\}$.

(a) Show that if $g : [0, \infty) \to \mathbb{R}$ is a continuous function of finite variation, then it has zero quadratic variation along any sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions such that $\lim_{n\to\infty} |\Pi_n| = 0$. (More generally, if g has finite p-variation, then it has zero r-variation for any $r > p$ along any sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions with $\lim_{n\to\infty} |\Pi_n| = 0$.)

(b) Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale null at 0. Show that if $[M] = 0$, then $M_t = 0$ P-a.s. for all $t \geq 0$.

Hint: Show the claim first when M is a square-integrable martingale. Extend then the conclusion by localisation.

(c) Show that a continuous local martingale $M = (M_t)_{t\geq 0}$ null at 0 and of finite variation is identically constant, i.e., $M_t = 0$ P-a.s. for all $t \geq 0$. Moreover, show that continuity is necessary, i.e., give an example of a local martingale $M = (M_t)_{t\geq 0}$ null at 0 of finite variation such that M is not identically equal to 0.

Hint: You may use the following result (compare with Theorem 4.1.4 in the lecture notes) to show that $[M] = 0$:

Let $M = (M_t)_{t\geq 0}$ be an RCLL local martingale null at 0. There exists a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions of $[0, \infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$ such that

$$P\left[\lim_{n\to\infty} V^2_t(M, \Pi_n) = [M]_t \text{ for all } t \geq 0\right] = 1.$$