Mathematical Foundations for Finance

Exercise Sheet 8

Exercise 8.1 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. On this space, let M be a local martingale null at 0 that satisfies $\sup_{0\leq t\leq T}|M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: You may use the dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Prove that

$$E\Big[[M]_t - [M]_s \,\Big|\, \mathcal{F}_s\Big] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s] \ P\text{-a.s., for } 0 \leq s \leq t \leq T.$$

Hint: You may use that $\mathrm{Var}[X \,|\, \mathcal{G}] = E\Big[(X - E[X \,|\, \mathcal{G}])^2 \,\Big|\, \mathcal{G}\Big]$.

Solution 8.1

(a) The process M is adapted by definition since it is a local martingale. Moreover, for any $s \in [0, T]$, it holds that

$$|M_s|^2 \leq |M_T^*|^2, \text{ where } M_T^* := \sup_{0\leq u\leq T}|M_u|.$$

By assumption, $M_T^* \in L^2$ and thus M is square-integrable on $[0, T]$.

Now, let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for M. For every fixed $n \in \mathbb{N}$, we have that

$$E[M_{\tau_n\wedge t} \,|\, \mathcal{F}_s] = M_{\tau_n\wedge s} \ P\text{-a.s., for } 0 \leq s \leq t \leq T, \tag{1}$$

because M is a local martingale. Since $|M_{\tau_n\wedge t}|$ is bounded from above by the integrable random variable M_T^*, for all $0 \leq t \leq T$, the dominated convergence theorem gives us that

$$\lim_{n\to\infty} E[M_{\tau_n\wedge t} \,|\, \mathcal{F}_s] = E\left[\lim_{n\to\infty} M_{\tau_n\wedge t} \,\middle|\, \mathcal{F}_s\right] = E[M_t \,|\, \mathcal{F}_s] \ \ P\text{-a.s.} \tag{2}$$

On the other hand, we have for the right-hand side of (1) that

$$\lim_{n\to\infty} M_{\tau_n\wedge s} = M_s \ P\text{-a.s.},$$

which, together with (2), gives us the martingale property for M on $[0, T]$ and concludes the proof.

(b) Since M is a square-integrable martingale on $[0,T]$, the square bracket process $[M]$ is integrable and $M^2 - [M]$ is a martingale according to Theorem 5.1.1 in the lecture notes. Therefore, for all $0 \le s \le t \le T$, it holds that

$$\begin{aligned}
E\Big[[M]_t - [M]_s \,\Big|\, \mathcal{F}_s\Big] &= E\left[M_t^2 - M_s^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\Big(M_t - E\left[M_t \,|\, \mathcal{F}_s\right]\Big)^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\Big(M_t - M_s + M_s - E\left[M_t \,|\, \mathcal{F}_s\right]\Big)^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\Big((M_t - M_s) - E\left[M_t - M_s \,|\, \mathcal{F}_s\right]\Big)^2 \,\middle|\, \mathcal{F}_s\right] \\
&= \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s] \ P\text{-a.s.}
\end{aligned}$$

Exercise 8.2 Let $(\Omega, \mathcal{F}, P)$ a probability space. We consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of square-integrable and independent random variables and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for all $k \in \mathbb{N}$. We assume that $(Y_k)_{k\in\mathbb{N}}$ are identically distributed, with $\mu := E\left[Y_k\right] \in \mathbb{R}$ and $\sigma^2 := \mathrm{Var}[Y_k] > 0$, for $k \in \mathbb{N}$. Define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by

$$X_n = \sum_{k=1}^{n} Y_k, \text{ for } n \in \mathbb{N}_0.$$

Note that X is adapted to $\mathbb{F}$ and integrable.

(a) Derive the Doob decomposition of X. In other words, find the martingale $M = (M_n)_{n\in\mathbb{N}_0}$ and the predictable and integrable process $A = (A_n)_{n\in\mathbb{N}_0}$ that are both null at zero and such that $X = M + A$ P-a.s. Deduce that M and A are square-integrable.
Hint: see Exercise 6.2(a).

(b) Find the optional quadratic variation $[M] = ([M]_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.
Hint: You may use Theorem 5.1.1 in the lecture notes, and in particular the condition $\Delta[M] = (\Delta M)^2$.

(c) Explicitly derive the predictable quadratic variation $\langle M\rangle = (\langle M\rangle_n)_{n\in\mathbb{N}_0}$ of the square-integrable martingale M.

Solution 8.2 To simplify notation, we omit "P-a.s." from all equalities below.

(a) Let us fix $n \in \mathbb{N}$. From Exercise 6.2(a), we know that

$$\begin{aligned} M_n &= \sum_{j=1}^{n} \Big(X_j - E\left[X_j \,|\, \mathcal{F}_{j-1} \right] \Big) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{j} \Big(Y_k - E\left[Y_k \,|\, \mathcal{F}_{j-1} \right] \Big) \\ &= \sum_{j=1}^{n} \Big(Y_j - E\left[Y_j \,|\, \mathcal{F}_{j-1} \right] \Big) \end{aligned}$$

since Y_k is $\mathcal{F}_{j-1}$-measurable for all $k \leq j-1$. Moreover, Y_j is independent of $\mathcal{F}_{j-1}$, and thus

$$M_n = \sum_{j=1}^{n} \Big(Y_j - E[Y_j] \Big) = X_n - n\mu.$$

Hence,

$$A_n = X_n - M_n = n\mu.$$

We conclude that both M and A are square-integrable since so is the process X by assumption.

(b) Since the process M is a square-integrable martingale, Theorem 5.1.1 from the lecture notes states that there exists a unique adapted increasing RCLL process $[M] = ([M]_n)_{n \in \mathbb{N}_0}$ null at 0 with $\Delta[M] = (\Delta M)^2$ and having the property that $M^2 - [M]$ is a local martingale. Hence, for each $n \in \mathbb{N}$, we have

$$\Delta[M]_n = (\Delta M_n)^2 = (M_n - M_{n-1})^2 = (Y_n - \mu)^2,$$

so that

$$[M]_n = \sum_{j=1}^{n} \Delta[M]_j = \sum_{j=1}^{n} (Y_j - \mu)^2,$$

(c) Since the process $[M]$ is integrable, we know there exists a unique increasing predictable and integrable process $\langle M \rangle = (\langle M \rangle_n)_{n \in \mathbb{N}_0}$ null at 0 such that $[M] - \langle M \rangle$ is a martingale. Thus, for each $n \in \mathbb{N}$, it holds that

$$E\Big[[M]_n - \langle M \rangle_n \,\Big|\, \mathcal{F}_{n-1} \Big] = [M]_{n-1} - \langle M \rangle_{n-1}.$$

The fact that $\langle M \rangle$ is predictable gives that

$$\begin{aligned} \langle M \rangle_n - \langle M \rangle_{n-1} &= E\left[[M]_n - [M]_{n-1} \,\Big|\, \mathcal{F}_{n-1} \right] \\ &= E\left[(Y_n - \mu)^2 \,\Big|\, \mathcal{F}_{n-1} \right] = \mathrm{Var}[Y_n] = \sigma^2, \end{aligned}$$

which in turn gives that $\langle M \rangle_n = n\sigma^2$.

Exercise 8.3 This exercise proves the frequently used fact that a continuous local martingale of finite variation is identically constant (and hence vanishes if it is null at 0).

For $p > 0$, the *(functional) p-variation* of a function $g : [0, \infty) \to \mathbb{R}$ is the function defined by

$$V^p(g) : [0,\infty) \to [0,\infty], \quad V_T^p(g) := \sup_\Pi V_T^p(g,\Pi) := \sup_\Pi \sum_{t_i \in \Pi} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p,$$

where the supremum is taken over all partitions Π of $[0, \infty)$, i.e., over all sets $\Pi \subseteq [0, \infty)$ with $0 \in \Pi$ and $\Pi \cap [0, t]$ finite for all $t \geq 0$. A function g has finite *(functional) p-variation* if $V_T^p(g) < \infty$ for all $T \geq 0$, and *finite (functional) variation* if it has finite (functional) 1-variation. For $p = 2$, we also say (functional) "quadratic variation" instead of "2-variation". We say that g has zero p-variation along a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions if $\lim_{n\to\infty} V_T^p(g, \Pi_n) = 0$ for all $T \geq 0$. For $\Pi := (t_i)_{i\in\mathbb{N}}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{N}$, we also define $|\Pi| := \sup\{t_{i+1} - t_i \mid t_i, t_{i+1} \in \Pi\}$.

(a) Show that if $g : [0, \infty) \to \mathbb{R}$ is a continuous function of finite variation, then it has zero quadratic variation along any sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions such that $\lim_{n\to\infty} |\Pi_n| = 0$. (More generally, if g has finite p-variation, then it has zero r-variation for any $r > p$ along any sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions with $\lim_{n\to\infty} |\Pi_n| = 0$.)

(b) Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale null at 0. Show that if $[M] = 0$, then $M_t = 0$ P-a.s. for all $t \geq 0$.

Hint: Show the claim first when M is a square-integrable martingale. Extend then the conclusion by localisation.

(c) Show that a continuous local martingale $M = (M_t)_{t\geq 0}$ null at 0 and of finite variation is identically constant, i.e., $M_t = 0$ P-a.s. for all $t \geq 0$. Moreover, show that continuity is necessary, i.e., give an example of a local martingale $M = (M_t)_{t\geq 0}$ null at 0 of finite variation such that M is not identically equal to 0.

Hint: You may use the following result (compare with Theorem 4.1.4 in the lecture notes) to show that $[M] = 0$:

Let $M = (M_t)_{t\geq 0}$ be an RCLL local martingale null at 0. There exists a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions of $[0, \infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$ such that

$$P\left[\lim_{n\to\infty} V_t^2(M, \Pi_n) = [M]_t \text{ for all } t \geq 0\right] = 1.$$

Solution 8.3

(a) Fix $T > 0$ and a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions with $\lim_{n\to\infty} |\Pi_n| = 0$. Since g is continuous, we have that $|g(t_i \wedge T) - g(t_{i-1} \wedge T)| \to 0$ as $|\Pi_n| \to 0$. But g is even uniformly continuous on the compact interval $[0, T]$, so we even have

$$\sup_{t_i \in \Pi_n} |g(t_i \wedge T) - g(t_{i-1} \wedge T)| \to 0 \quad \text{as } |\Pi_n| \to 0.$$

Then we have for any n

$$\begin{aligned} V_T^r(g, \Pi_n) &= \sum_{t_i \in \Pi_n} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^r \\ &\le \sup_{t_i \in \Pi_n} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^{r-p} \sum_{t_i \in \Pi_n} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p \\ &\le \sup_{t_i \in \Pi_n} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^{r-p} \sup_{\Pi} \sum_{t_i \in \Pi} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p . \end{aligned}$$

The second factor is $V_T^p(g) < \infty$ by assumption, and the first factor goes to 0 as $n \to \infty$.

(b) By Theorem 5.1.1 in the lecture notes, $M^2 - [M] = M^2$ is a local martingale null at 0. Thus, there exists a localising sequence $(\tau_n)_n$ such that $(M^{\tau_n})^2$ is a martingale null at 0. Let $\widetilde{M} = M^{\tau_n}$. Then we have

$$E\left[\widetilde{M}_t^2\right] = 0 \quad \text{for} \quad t \ge 0\,.$$

Therefore, $\widetilde{M}_t = 0$ P-a.s. and hence $M_t^{\tau_n} = 0$ P-a.s. for all $n \in \mathbb{N}$ and all $t \ge 0$. Letting $n \to \infty$, we obtain $M_t = 0$ P-a.s. for all $t \ge 0$.

(c) Let M be a continuous local martingale null at 0 which has paths of finite variation. By the hint, we conclude that for a well-chosen sequence $(\Pi_n)_{n \in \mathbb{N}}$ of partitions such that $|\Pi_n| \to 0$ as $n \to \infty$, we have that

$$P\left[\lim_{n \to \infty} V_T^2(M, \Pi_n) = [M]_T \quad \text{for all } T \ge 0\right] = 1.$$

But by assumption, the paths of M, i.e., the functions $t \mapsto M_t(\omega)$, are continuous and of finite variation. Hence by part a), $\lim_{n \to \infty} V_T^2(M(\omega), \Pi_n) = 0$ for all those $\omega \in \Omega$ for which $t \mapsto M_t(\omega)$ is of finite variation. By definition, this is a set of full probability and hence $t \mapsto [M]_t(\omega)$ is identically equal to 0 for P-a.a. ω. Part b) then implies the claim.

Define

$$M_t := \begin{cases} 0 & \text{for } t < 1 \\ Z & \text{for } t \ge 1, \end{cases}$$

where Z is any integrable $\mathcal{F}_t$-measurable random variable with $E[Z] = 0$. Since M is of finite variation, if $Z \ne 0$ a.s., then $M = (M_t)_{t \ge 0}$ is a counterexample that shows that (a.s.) continuity is necessary.