Mathematical Foundations for Finance

Exercise Sheet 9

Please hand in your solutions by 12:00 on Wednesday, November 27 via the course homepage.

Exercise 9.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) For some constants $S_0 > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the *geometric Brownian motion* $S = (S_t)_{t\geqslant 0}$ as follows

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Compute $\lim_{t\to\infty} S_t$ when $\mu \neq \frac{\sigma^2}{2}$. Determine whether the limit exists if $\mu = \frac{\sigma^2}{2}$.
Hint: You may use the law of the iterated logarithm.

(b) Prove that

$$E[W_t^3 - W_s^3 \mid \mathcal{F}_s] = 3(t-s)W_s \ P\text{-a.s., for } 0 \leq s < t.$$

Hint: You may compute $E[(W_t - W_s)^3 \mid \mathcal{F}_s]$.

Exercise 9.2 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted process $X = (X_t)_{t\geq 0}$ null at 0. Assume that X is integrable and has independent and stationary increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for all $t > s \geq 0$.

(a) Under which conditions on $(E[X_t])_{t\geq 0}$ is X a martingale? And a supermartingale? A submartingale?

(b) From this point onward, let us assume that X is a square-integrable martingale. Prove that

$$E[X_t^2] + E[X_s^2] = E[X_{t+s}^2] \text{ for any } t, s \geq 0,$$

and deduce that $(E[X_t^2])_{t\geq 0}$ is an increasing process.

(c) Deduce from (b) that $E[X_t^2] = tE[X_1^2]$ for all $t \geq 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Then, deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.

(d) Prove that $\langle X\rangle_t = tE\left[X_1^2\right]$ for all $t \geq 0$.
Hint: You may use your result from (c).

Exercise 9.3 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F}$ satisfies the usual conditions.

(a) Let X be an adapted process and τ a stopping time. Show that if X^τ is a martingale, then so is X^σ for any stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You may use the result that a stopped martingale is again a martingale.

(b) Let M and N be two local martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is a local martingale.
Hint: You may use your result in (a).

(c) We say that two Brownian motions W^1 and W^2 on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are *correlated with instantaneous correlation* $\rho \in [-1, 1]$ if, for $s \leq t$, the increments $W_t^1 - W_s^1$ and $W_t^2 - W_s^2$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $\left[W^1, W^2\right]_t = \rho t$ P-a.s.
Hint: You may find $\lambda \in \mathbb{R}$ such that $B^\lambda := \lambda(W^1 + W^2)$ is a Brownian motion. Then, compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\,\cdot\,]$.