Mathematical Foundations for Finance

Exercise Sheet 9

Exercise 9.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) For some constants $S_0 > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the *geometric Brownian motion* $S = (S_t)_{t\geqslant 0}$ as follows

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Compute $\lim_{t\to\infty} S_t$ when $\mu \neq \frac{\sigma^2}{2}$. Determine whether the limit exists if $\mu = \frac{\sigma^2}{2}$.
Hint: You may use the law of the iterated logarithm.

(b) Prove that

$$E[W_t^3 - W_s^3 \mid \mathcal{F}_s] = 3(t-s)W_s \ P\text{-a.s., for } 0 \leq s < t.$$

Hint: You may compute $E[(W_t - W_s)^3 \mid \mathcal{F}_s]$.

Solution 9.1

(a) For any $t \geq 0$, we can rewrite S_t as

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{2t\log\log t}\frac{W_t}{\sqrt{2t\log\log t}}\right) \\ &= S_0 \exp\left(\sqrt{2t\log\log t}\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{t}{\sqrt{2t\log\log t}} + \sigma\frac{W_t}{\sqrt{2t\log\log t}}\right)\right). \end{aligned}$$

Since

$$\lim_{t\to\infty} \sqrt{2t\log\log t} = +\infty \quad \text{and} \quad \lim_{t\to\infty} \frac{t}{\sqrt{2t\log\log t}} = +\infty,$$

and by the law of the iterated logarithm, it follows that:

- when $\mu > \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = +\infty \ P\text{-a.s.};$$
- when $\mu < \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = 0 \ P\text{-a.s.}$$

If $\mu = \frac{\sigma^2}{2}$, the limit $\lim_{t\to\infty} S_t$ does not exist since

$$P\Big(\liminf_{t\to\infty} S_t = 0\Big) = 1 \quad \text{and} \quad P\Big(\limsup_{t\to\infty} S_t = +\infty\Big) = 1.$$

(b) To simplify notation, we omit "P-a.s." from all equalities below. Let us fix some $0 \le s < t$. Since $W_t - W_s$ is independent of $\mathcal{F}_s$, so is $(W_t - W_s)^3$. Hence,

$$E[(W_t - W_s)^3 \mid \mathcal{F}_s] = E[(W_t - W_s)^3].$$

Since $W_t - W_s \sim N(0, t-s)$, then $E[(W_t - W_s)^3] = 0$, and thus

$$\begin{aligned} E[W_t^3 - W_s^3 \mid \mathcal{F}_s] &= E[W_t^3 - W_s^3 \mid \mathcal{F}_s] - E[(W_t - W_s)^3 \mid \mathcal{F}_s] \\ &= E[3W_t^2 W_s - 3W_t W_s^2 \mid \mathcal{F}_s]. \end{aligned}$$

From the fact that W_t and W_s are normal random variables, we deduce that W_t, W_t^2, W_s, W_s^2, and all products, are integrable. Hence, we get

$$\begin{aligned} E[W_t^3 - W_s^3 \mid \mathcal{F}_s] &= 3W_s E[W_t^2 \mid \mathcal{F}_s] - 3W_s^2 E[W_t \mid \mathcal{F}_s] \\ &= 3W_s(W_s^2 + t - s) - 3W_s^3 \\ &= 3(t-s)W_s, \end{aligned}$$

where in the second step we have used that $(W)_{t\geqslant 0}$ and $(W_t^2 - t)_{t\geqslant 0}$ are martingales.

Exercise 9.2 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted process $X = (X_t)_{t\ge 0}$ null at 0. Assume that X is integrable and has independent and stationary increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for all $t > s \ge 0$.

(a) Under which conditions on $(E[X_t])_{t\ge 0}$ is X a martingale? And a supermartingale? A submartingale?

(b) From this point onward, let us assume that X is a square-integrable martingale. Prove that

$$E[X_t^2] + E[X_s^2] = E[X_{t+s}^2] \text{ for any } t, s \ge 0,$$

and deduce that $(E[X_t^2])_{t\ge 0}$ is an increasing process.

(c) Deduce from (b) that $E[X_t^2] = tE[X_1^2]$ for all $t \ge 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Then, deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.

(d) Prove that $\langle X\rangle_t = tE[X_1^2]$ for all $t \ge 0$.
Hint: You may use your result from (c).

Solution 9.2

(a) Adaptedness and integrability are already given by assumption. Let us fix some $t > s \geq 0$. We can then use the fact that X has independent and stationary increments to compute

$$E[X_t \,|\, \mathcal{F}_s] = E[X_t - X_s \,|\, \mathcal{F}_s] + X_s = E[X_t - X_s] + X_s = E[X_{t-s}] + X_s \ P\text{-a.s.}$$

As a result, X is a martingale if and only if $E[X_t] = 0$ for all $t \geq 0$, a supermartingale if and only if $E[X_t] \leq 0$ for all $t \geq 0$, and a submartingale if and only if $E[X_t] \geq 0$ for all $t \geq 0$.

(b) Let us fix $t, s > 0$. By the martingale property of X and the stationarity of the increments, we can directly compute

$$E[X_{t+s}^2] - E[X_t^2] = E[X_{t+s}^2 - X_t^2] = E[(X_{t+s} - X_t)^2] = E[X_s^2]$$

as a consequence of Exercise 9.1. Consequently, we have that

$$E[X_t^2] - E[X_s^2] = E[X_{t-s}^2] \geq 0 \text{ for any } t \geq s,$$

proving that the process $(E[X_t^2])_{t\geq 0}$ is increasing.

(c) Let $t = 1/n$ for some $n \in \mathbb{N}$. We want to show that $nE[X_{1/n}^2] = E[X_1^2]$. It holds that

$$\begin{aligned} nE[X_{1/n}^2] = \sum_{k=1}^{n} E[X_{1/n}^2] &= \sum_{k=1}^{n} \Big(E[X_{k/n}^2] - E[X_{(k-1)/n}^2] \Big) \\ &= E[X_1^2] - E[X_0^2] = E[X_1^2], \end{aligned}$$

where in the second equality we have used our result from (b). If we now consider an arbitrary number $\ell/n \in \mathbb{Q}_+$, we can use the same technique as in the above to compute

$$\ell E[X_{1/n}^2] = \sum_{k=1}^{\ell} E[X_{1/n}^2] = E[X_{\ell/n}^2].$$

Since $E[X_{1/n}^2] = E[X_1^2]/n$, we can conclude that $E[X_{\ell/n}^2] = \ell E[X_1^2]/n$. Therefore, we have proved that $E[X_t^2] = tE[X_1^2]$ for all $t \in \mathbb{Q}_+$. We can conclude the proof using the fact that $(E[X_t^2])_{t\geq 0}$ is increasing. Precisely,

$$\begin{aligned} tE[X_1^2] = \sup_{s\in\mathbb{Q}_+, s<t} sE[X_1^2] &= \sup_{s\in\mathbb{Q}_+, s<t} E[X_s^2] \\ &\leq E[X_t^2] \leq \inf_{s\in\mathbb{Q}_+, s>t} E[X_s^2] = tE[X_1^2], \end{aligned}$$

and thus conclude that $E[X_t^2] = tE[X_1^2]$.

(d) We first have to show that $(tE[X_1^2])_{t\geq 0}$ is an increasing, predictable process null at 0. Since $E[X_1^2] \geq 0$, the process is clearly increasing. Moreover, it is deterministic and continuous, thus it is predictable and it is clearly null at 0. It only remains to show that the process $(X_t^2 - tE[X_1^2])_{t\geq 0}$ is a martingale. Since the increments are independent and stationary we can compute

$$\begin{aligned} E[X_t^2 - X_s^2|\mathcal{F}_s] = E[(X_t - X_s)^2|\mathcal{F}_s] &= E[(X_t - X_s)^2] \\ &= E[X_{t-s}^2] \\ &= (t-s)E[X_1^2] \\ &= tE[X_1^2] - sE[X_1^2] \ P\text{-a.s. for all } t > s \geq 0, \end{aligned}$$

where the fourth equality uses our result from (c). Rearranging the above, we obtain that

$$E[X_t^2 - tE[X_1^2] - (X_s^2 - sE[X_1^2])|\mathcal{F}_s] = 0 \ P\text{-a.s.},$$

which is the martingale property for $(X_t^2 - tE[X_1^2])_{t\geq 0}$.

Exercise 9.3 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F}$ satisfies the usual conditions.

(a) Let X be an adapted process and τ a stopping time. Show that if X^τ is a martingale, then so is X^σ for any stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You may use the result that a stopped martingale is again a martingale.

(b) Let M and N be two local martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is a local martingale.
Hint: You may use your result in (a).

(c) We say that two Brownian motions W^1 and W^2 on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are *correlated with instantaneous correlation* $\rho \in [-1, 1]$ if, for $s \leq t$, the increments $W_t^1 - W_s^1$ and $W_t^2 - W_s^2$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P-a.s.
Hint: You may find $\lambda \in \mathbb{R}$ such that $B^\lambda := \lambda(W^1 + W^2)$ is a Brownian motion. Then, compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\,\cdot\,]$.

Solution 9.3

(a) For notational clarity, we define $Y := X^\tau$. Note that since $\sigma \leq \tau$ P-a.s. by assumption, we can write for all $t \geq 0$ that

$$X_t^\sigma = X_{t\wedge\sigma} = X_{t\wedge\tau\wedge\sigma} = X_{t\wedge\sigma}^\tau = Y_t^\sigma \ P\text{-a.s.}$$

But Y is a martingale by assumption; so Y^σ, the stopped martingale, is a martingale as well. The above equation then directly implies the same for X^σ.

(b) Let $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be two localizing sequences for M and N, respectively. Let us fix $n \in \mathbb{N}$, and define $\theta_n := \min(\tau_n, \sigma_n)$. It follows that $\theta_n \le \tau_n$ P-a.s. as well as $\theta_n \le \sigma_n$ P-a.s., and thus our result from (a) implies that both M^{θ_n} and N^{θ_n} are martingales if θ_n is a stopping time. We can conclude that

$$\alpha M^{\theta_n} + \beta N^{\theta_n} = (\alpha M + \beta N)^{\theta_n} \ P\text{-a.s.}$$

is a martingale.

What thus remains to be shown is that $(\theta_n)_{n\in\mathbb{N}}$ is indeed a sequence of stopping times with $\theta_n \nearrow \infty$ P-a.s. The fact that $\theta_n \nearrow \infty$ P-a.s. is trivial since we have that both $\tau_n \nearrow \infty$ and $\sigma \nearrow \infty$ P-a.s. In order to show that θ_n is a stopping time for each $n \in \mathbb{N}$, we note that

$$\{\theta_n \le t\} = \{\min(\tau_n, \sigma_n) \le t\} = \{\tau_n \le t\} \cup \{\sigma_n \le t\} \in \mathcal{F}_t,$$

since $\{\tau_n \le t\} \in \mathcal{F}_t$ and $\{\sigma_n \le t\} \in \mathcal{F}_t$ because τ_n and σ_n are stopping times. This shows that $(\theta_n)_{n\in\mathbb{N}}$ is a localizing sequence for $\alpha M + \beta N$ and concludes the proof.

(c) Let us define $B^\lambda := \lambda(W^1 + W^2)$, for $\lambda \in \mathbb{R}$. The process B^λ is adapted and such that $B^\lambda_0 = 0$ P-a.s., and its trajectories are continuous for P-a.e. $\omega \in \Omega$. Therefore, for it to be a Brownian motion, we need to check that $B^\lambda_t - B^\lambda_s$ is independent of $\mathcal{F}_s$ and has a normal distribution $\mathcal{N}(0, t-s)$, for any $0 \le s \le t$. We have that

$$\begin{aligned} B^\lambda_t - B^\lambda_s = \lambda(W^1_t - W^1_s) + \lambda(W^2_t - W^2_s) &\sim \mathcal{N}(0, \lambda^2(2(t-s) + 2\rho(t-s))) \\ &\sim \mathcal{N}(0, \lambda^2(t-s)(2+2\rho)) \end{aligned}$$

because W^1 and W^2 are Brownian motions such that $(W^1_t - W^1_s, W^2_t - W^2_s) \sim \mathcal{N}(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix},$$

and we know that linear transformations of normal random vectors are normally distributed. We can deduce that by setting $\lambda^2 = 1/(2+2\rho)$, B^λ is a Brownian motion.

As suggested in the hint, let us now compute the quadratic variation of B^λ using that $[B^\lambda] = [B^\lambda, B^\lambda]$ and the bilinearity and symmetry of $[\,\cdot\,,\,\cdot\,]$. We have that

$$\begin{aligned}[B^\lambda]_t &= [\lambda(W^1+W^2), \lambda(W^1+W^2)]_t = \lambda^2[W^1+W^2, W^1+W^2]_t \\ &= \lambda^2([W^1, W^1+W^2]_t + [W^2, W^1+W^2]_t) \\ &= \lambda^2([W^1, W^1]_t + [W^1, W^2]_t + [W^2, W^1]_t + [W^2, W^2]_t) \\ &= \lambda^2([W^1]_t + 2[W^1, W^2]_t + [W^2]_t) \\ &= 2\lambda^2([W^1, W^2]_t + t),\end{aligned}$$

where the last equality follows from the fact that W^1 and W^2 are Brownian motions, and we thus have that $[W^1]_t = t$ and $[W^2]_t = t$ P-a.s. We can thus rearrange the above terms to obtain that

$$[W^1, W^2]_t = \frac{1}{2\lambda^2}[B^\lambda]_t - t.$$

But the choice $\lambda^2 = 1/(2+2\rho)$ leads to B^λ being a Brownian motion, in which case $[B^\lambda]_t = t$ P-a.s. We conclude that

$$[W^1, W^2]_t = t\left(\frac{1}{2\lambda^2} - 1\right) = t\left(\frac{2+2\rho}{2} - 1\right) = \rho t \ P\text{-a.s. for all } t \geq 0.$$