Exercise Sheet 2

Localization, local rings, radicals

Exercises 1, 2, 4 and 6 are taken from the book *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. Let R be a ring, let S and T be two multiplicative subsets of R, and let U be the image of T in $S^{-1}R$. Show that the rings $(ST)^{-1}R$ and $U^{-1}(S^{-1}R)$ are isomorphic.

2. Let $\varphi\colon R \to R'$ be a ring homomorphism. Prove for any ideals $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}$ of R and any ideals $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}$ of R':

$$\begin{array}{ll}
\varphi_*(\mathfrak{a}_1 + \mathfrak{a}_2) = \varphi_*(\mathfrak{a}_1) + \varphi_*(\mathfrak{a}_2) & \varphi^*(\mathfrak{b}_1 + \mathfrak{b}_2) \supset \varphi^*(\mathfrak{b}_1) + \varphi^*(\mathfrak{b}_2) \\
\varphi_*(\mathfrak{a}_1 \cap \mathfrak{a}_2) \subset \varphi_*(\mathfrak{a}_1) \cap \varphi_*(\mathfrak{a}_2) & \varphi^*(\mathfrak{b}_1 \cap \mathfrak{b}_2) = \varphi^*(\mathfrak{b}_1) \cap \varphi^*(\mathfrak{b}_2) \\
\varphi_*(\mathfrak{a}_1 \cdot \mathfrak{a}_2) = \varphi_*(\mathfrak{a}_1) \cdot \varphi_*(\mathfrak{a}_2) & \varphi^*(\mathfrak{b}_1 \cdot \mathfrak{b}_2) \supset \varphi^*(\mathfrak{b}_1) \cdot \varphi^*(\mathfrak{b}_2) \\
\varphi_*(\mathrm{Rad}(\mathfrak{a})) \subset \mathrm{Rad}(\varphi_*(\mathfrak{a})) & \varphi^*(\mathrm{Rad}(\mathfrak{b})) = \mathrm{Rad}(\varphi^*(\mathfrak{b}))
\end{array}$$

   The set of ideals $\{\varphi_*\mathfrak{a} \mid \mathfrak{a} \subset R \text{ ideal}\}$ in R' is closed under sum and product, and the set of ideals $\{\varphi^*\mathfrak{b} \mid \mathfrak{b} \subset R' \text{ ideal}\}$ in R is closed under the other two operations.

3. (a) For an integer $a > 1$ and a non-zero integer b, consider the multiplicative subset S of $\mathbb{Z}/a\mathbb{Z}$ consisting of the residue classes $b^n + a\mathbb{Z}$ for all $n \geqslant 0$. Compute $S^{-1}(\mathbb{Z}/a\mathbb{Z})$.

   (b) Compute the localization $S^{-1}R$ for the ring $R = K[X,Y]/(XY)$ and the multiplicative set $S = \{X^n \mid n \geqslant 0\}$.

   (c) Show that for any ring R and any $t \in R$ the localization R_t of R at t is isomorphic to $R[Y]/(tY - 1)$.

4. Let R be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $R_\mathfrak{p}$ has no nilpotent element $\neq 0$.

   (a) Show that R has no nonzero nilpotent element.

   (b) If each $R_\mathfrak{p}$ is an integral domain, is R necessarily an integral domain?

5. Prove that for any ideal $\mathfrak{a} \subset R$, we have $\mathrm{Rad}(\mathfrak{a})/\mathfrak{a} = \mathrm{rad}(R/\mathfrak{a})$.

*6. For every element f of a ring R let D_f denote the complement of $V(f)$ in $X :=$ Spec R. The sets D_f are called *basic open sets* of X. Show that

(a) The basic open sets form a basis of the Zariski topology.

(b) $D_f \cap D_g = D_{fg}$.

(c) $D_f = \varnothing$ if and only if f is nilpotent.

(d) $D_f = X$ if and only if f is a unit.

(e) $D_f = D_g$ if and only if $\operatorname{Rad}((f)) = \operatorname{Rad}((g))$.

(f) X is quasi-compact, i.e., every open cover has a finite subcover.