Mathematical Finance

Exercise Sheet 1

Submit by 12:00 on Wednesday, October 2 via the course homepage.

Exercise 1.1 *(Path regularity and measurability)* Let $S = (S_t)_{t\geqslant 0}$ be a real-valued stochastic process. Define the processes S^* and A by $S^*_t := \sup_{0\leqslant r\leqslant t} S_r$ and $A_t := \int_0^t S_r \,\mathrm{d}r$ (when it exists), respectively.

(a) Show that if S is RCLL, then S^* is RCLL and A is well defined and continuous.

Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions.

(b) Show that if S is RCLL and adapted, then also S^* and A are adapted.

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and define the process $\vartheta = (\vartheta_t)_{t\geqslant 0}$ by $\vartheta_t := f(S_t, S^*_t, A_t)$.

Show that if S is adapted and continuous, then ϑ is predictable.

Exercise 1.2 *(Reparametrisation)* Fix a finite time horizon $T > 0$ and let $S = (S_t)_{0\leqslant t\leqslant T}$ be a semimartingale. Prove that there is a bijection between self-financing strategies $\varphi = (\varphi^0, \vartheta)$ and pairs

$$(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}.$$

Give explicitly the bijection map and its inverse.

Exercise 1.3 *(Brownian motion)* Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ be a filtered probability space satisfying the usual conditions, and let $B = (B^1_t, B^2_t, B^3_t)_{t\geq 0}$ be a three-dimensional Brownian motion starting at $(1, 0, 0)$. Define the process $X = (X_t)_{t\geq 0}$ by $X_t := \|B_t\|$, where $\|\cdot\|$ denotes the Euclidean norm.

(a) Let $a, b > 0$ such that $a < 1 < b$, and consider the stopping time

$$\tau_{a,b} := \inf\{t \geq 0 : \ X_t \leq a \text{ or } X_t \geq b\}.$$

Show that $Y = (Y_t)_{t\geq 0}$ defined by $Y_t := (X_{\tau_{a,b}\wedge t})^{-1}$ is a bounded martingale.

(b) Show that $P\Big[X_t = 0 \text{ for some } t \geq 0\Big] = 0$.

(c) Show that X satisfies the SDE

$$\mathrm{d}X_t = \frac{1}{X_t}\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 1, \tag{1}$$

where $W = (W_t)_{t \geq 0}$ is a one-dimensional Brownian motion starting at 0.

(d) Let $(S, 1) = (S_t, 1)_{t \in [0,1]}$ be a continuous-time model with time horizon $T = 1$, where S satisfies the SDE

$$\mathrm{d}S_t = S_t \left(\left(\frac{1}{X_t} + 2 \right) \mathrm{d}t + \mathrm{d}W_t \right), \quad S_0 = 1,$$

and where W is the Brownian motion introduced in (1). Show that S fails the NA condition by showing that the strategy $\theta = (\theta_t)_{t \in [0,1]}$, defined by $\theta_t := (S_t)^{-1}$, is an arbitrage opportunity.

Exercise 1.4 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t \geqslant 0}$ by

$$S_t := S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

The process $S = (S_t)_{t \geqslant 0}$ is called a *geometric Brownian motion* and is the stock price process in the *Black–Scholes model*.

Find $\lim_{t \to \infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You may use the law of the iterated logarithm.