Mathematical Finance

Exercise Sheet 12

Submit by 12:00 on Wednesday, December 18 via the course homepage.

Exercise 12.1 *(Some properties of u)* Let $U : (0, \infty) \to \mathbb{R}$ be a concave and increasing function. Define the function $u : (0, \infty) \to (-\infty, +\infty]$ by

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)],$$

where $\mathcal{V}(x) := \{x + G(\vartheta) : \vartheta \in \Theta^x_{\text{adm}}\}$.

(a) Show that u is concave and increasing.

(b) If additionally $u(x_0) < \infty$ for some $x_0 > 0$, show that $u(x) < \infty$ for all $x > 0$.

Exercise 12.2 *(Utility in a market with arbitrage)* Consider a general market with finite time horizon T. Let $U : (0, \infty) \to \mathbb{R}$ be an increasing and concave utility function. Suppose that U is unbounded from above and that either the market admits a 0-admissible arbitrage opportunity, or we are in finite discrete time and the market admits an (admissible) arbitrage opportunity. Show that in both cases, we have $u \equiv \infty$.

Without imposing that U is unbounded from above, what can you say about the relationship between $u(x)$ and $U(x)$ as $x \to \infty$?

Exercise 12.3 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function such that $u(x) < \infty$ for some (and hence for all) $x \in (0, \infty)$. Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial.

(a) Show that $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. for all $h \in \mathcal{D}(z)$, and deduce that

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

(b) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j defined in the lecture notes is in $C^1((z_0, \infty); \mathbb{R})$ and satisfies

$$j'(z) = E\left[\frac{\mathrm{d}Q}{\mathrm{d}P} J'\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right], \quad z \in (z_0, \infty).$$

(c) Set $x_0 := \lim_{z \downarrow z_0}(-j'(z))$ and fix $x \in (0, x_0)$. Let $z_x \in (z_0, \infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I(z_x \frac{\mathrm{d}Q}{\mathrm{d}P})$ is the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)].$$

Exercise 12.4 *(The Merton problem)* Consider the Black–Scholes market given by

$$\begin{aligned} \mathrm{d}\widetilde{S}^0_0 &= r\widetilde{S}^0_t \,\mathrm{d}t, & \widetilde{S}^0_0 &= 1, \\ \mathrm{d}\widetilde{S}^1_t &= \widetilde{S}^1_t(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t), & \widetilde{S}^1_0 &= s > 0. \end{aligned}$$

Let $U : (0, \infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma} x^\gamma$, where $\gamma \in (-\infty, 1)\backslash\{0\}$. We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$).

(a) Show that for $z > 0$,

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \frac{(\mu - r)^2 T}{\sigma^2}\right).$$

(b) Show that the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)], \quad x \in (0, \infty),$$

is given by $f^*_x := x\mathcal{E}(\frac{1}{1-\gamma} \frac{\mu - r}{\sigma} R)_T$, where the process $R = (R_t)_{0 \leqslant t \leqslant T}$ is defined by $R_t = W_t + \frac{\mu - r}{\sigma} t$.

(c) Deduce that $f^*_x = V_T(x, \vartheta^x)$, where the integrand $\vartheta^x = (\vartheta^x_t)_{0 \leqslant t \leqslant T}$ is given by

$$\vartheta^x_t = \frac{x}{S^1_t} \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2} \mathcal{E}\left(\frac{1}{1-\gamma} \frac{\mu - r}{\sigma} R\right)_t, \quad x \in (0, \infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2} \frac{\gamma}{1-\gamma} \frac{(\mu - r)^2}{\sigma^2} T\right), \quad x \in (0, \infty).$$

(d) For any x-admissible ϑ with $V(x, \vartheta) > 0$, denote by

$$\pi_t := \frac{\vartheta_t S^1_t}{V_t(x, \vartheta)}$$

the fraction of wealth that is invested in the stock. Show that the optimal strategy ϑ^x is given by the *Merton proportion*

$$\pi^*_t = \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2}.$$

Exercise 12.5 *($\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ has moments of all orders)* Let S be a continuous real-valued semimartingale satisfying the structure condition (SC), i.e. there exist a continuous local martingale M null at zero and a predictable process λ such that

$$S = S_0 + M + \int \lambda \,\mathrm{d}\langle M \rangle,$$

and with the mean-variance tradeoff process $K = \int \lambda^2 \,\mathrm{d}\langle M \rangle$ bounded. Now define $\hat{Z} := \mathcal{E}(-\lambda \bullet M)$ and $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \hat{Z}_T$.

(a) Show that $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}(S)$.

(b) Show that both $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $\frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$ have moments of all orders.