Mathematical Finance

Exercise Sheet 3

Submit by 12:00 on Wednesday, October 16 via the course homepage.

Exercise 3.1 *(Integrability property of local martingales)* Fix a finite time horizon $T > 0$ and define the space $\mathcal{H}^1$ of martingales by

$$\mathcal{H}^1 := \left\{ M = (M_t)_{0 \leqslant t \leqslant T} : M \text{ RCLL martingale, } M_T^* := \sup_{0 \leqslant t \leqslant T} |M_t| \in L^1 \right\}.$$

Show that every (RCLL) local martingale is locally in $\mathcal{H}^1$. That is, for each local martingale M, show that there is a sequence of stopping times $\tau_n \uparrow T$ stationarily such that $M^{\tau_n} \in \mathcal{H}^1$ for all $n \in \mathbb{N}$.

Exercise 3.2 *(Doob decomposition)* Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$ be a filtered probability space in discrete time, and let $X = (X_k)_{k \in \mathbb{N}_0}$ be a supermartingale.

(a) Prove that there exist a martingale $M = (M_k)_{k \in \mathbb{N}_0}$ and an increasing, integrable and predictable process $A = (A_k)_{k \in \mathbb{N}_0}$ such that

$$X = X_0 + M - A.$$

(b) Prove that if we further impose that M and A are both null at zero, then they are unique up to P-a.s. equality.

Exercise 3.3 *(Conditional expectation of increments)* Let $X = (X_t)_{t \geqslant 0}$ be an adapted bounded RCLL process. Prove that for each fixed $t \geqslant 0$,

$$\lim_{u \downarrow t} E[X_u - X_t \mid \mathcal{F}_t] = 0.$$

Can we relax boundedness to a weaker condition?

What can we say for $\lim_{s \uparrow t} E[X_t - X_s \mid \mathcal{F}_s]$, where again $t > 0$ is fixed?