

Mathematical Finance

Exercise Sheet 4

Submit by 12:00 on Wednesday, October 23 via the course homepage.

Exercise 4.1 Fix a finite time horizon $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, and an adapted RCLL process $S = (S_t)_{0 \leq t \leq T}$. Recall that \mathfrak{bE} denotes the space of “simple integrands” ϑ , i.e., processes of the form

$$\vartheta = \sum_{i=1}^n h_i \mathbf{1}_{\llbracket \tau_{i-1}, \tau_i \rrbracket},$$

where $n \in \mathbb{N}$, $0 \leq \tau_0 \leq \tau_1 \leq \dots \leq \tau_n \leq T$ are stopping times, and h_i are \mathbb{R}^d -valued, bounded and $\mathcal{F}_{\tau_{i-1}}$ -measurable random variables. For $\vartheta \in \mathfrak{bE}$, we have

$$G_t(\vartheta) = \sum_{i=1}^n h_i (S_{\tau_i \wedge t} - S_{\tau_{i-1} \wedge t}).$$

In the lecture, we have seen that the existence of an equivalent local martingale measure (ELMM) guarantees $\text{NA}_{\text{elem}}^{\text{adm}}$. However, the example below illustrates that if we remove the admissibility constraint, this result fails; that is, the existence of an ELMM does not guarantee NA_{elem} holds.

Let $W = (W_t)_{0 \leq t \leq T}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and define S by

$$S_t := \begin{cases} \frac{1}{\sqrt{2(T-t)}} \exp\left(-\frac{W_t^2}{2(T-t)}\right) & \text{if } 0 \leq t < T, \\ 0 & \text{if } t = T. \end{cases}$$

- (a) Show that S is a continuous local P -martingale.
- (b) Show that S is not a P -martingale.
- (c) Construct a *non-admissible* “one-step buy-and-hold” simple strategy which gives an arbitrage opportunity.

Exercise 4.2 Fix a finite time horizon $T > 0$, a semimartingale $S = (S_t)_{0 \leq t \leq T}$ and a simple integrand $\vartheta \in \mathfrak{bE}$ with

$$\vartheta = \sum_{i=1}^n h_i \mathbf{1}_{\llbracket \tau_{i-1}, \tau_i \rrbracket},$$

where $n \in \mathbb{N}$, $0 \leq \tau_0 \leq \tau_1 \leq \dots \leq \tau_n \leq T$ are stopping times, and h_i are \mathbb{R}^d -valued, bounded, and $\mathcal{F}_{\tau_{i-1}}$ -measurable random variables. Suppose moreover that $G_T(\vartheta) \geq 0$ P -a.s.

Consider the following two statements (a) and (b).

- (a) Suppose that $G_T(\vartheta) \neq 0$. Then there exists $\vartheta' \in \mathbf{bE}$ with $\vartheta' = \sum_{k=1}^m h'_k \mathbf{1}_{\llbracket \tau'_{k-1}, \tau'_k \rrbracket}$ and $G_T(\vartheta') \geq 0$ P -a.s. as well as $G_T(\vartheta') \neq 0$ and such that ϑ' is 0-admissible in discrete time, in the sense that $G_{\tau'_k}(\vartheta') \geq 0$ P -a.s. for all k .
- (b) Suppose S admits an ELMM Q . By using the DMW theorem in the discrete-time model with (random) trading dates τ_0, \dots, τ_n , we deduce that $G_T(\vartheta) = 0$ P -a.s.

Prove or disprove (a) and (b). If things go wrong, identify where as precisely as possible. What changes in (b) if Q is an EMM?

Exercise 4.3 Suppose that $f, g : [0, T] \rightarrow \mathbb{R}$ are functions of finite variation. Establish the integration by parts formulas

$$\begin{aligned} f(T)g(T) - f(0)g(0) &= \int_0^T f(s) dg(s) + \int_0^T g(s-) df(s) \\ &= \int_0^T f(s-) dg(s) + \int_0^T g(s) df(s) \\ &= \int_0^T f(s-) dg(s) + \int_0^T g(s-) df(s) + \sum_{0 < s \leq T} \Delta f(s) \Delta g(s). \end{aligned}$$