Coordinator
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Mathematical Finance

Exercise Sheet 5

Submit by 12:00 on Wednesday, October 30 via the course homepage.

Exercise 5.1 *(Convergence in probability)* Consider the metric d on L^0 defined by $d(X, Y) := E[1 \wedge |X - Y|]$. Show that for $X_n, X \in L^0$, we have

$$X_n \to X \text{ in probability} \quad \Longleftrightarrow \quad d(X_n, X) \to 0.$$

Exercise 5.2 *(Good integrator)* Fix a finite time horizon $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$, and an adapted RCLL process $X = (X_t)_{0 \leqslant t \leqslant T}$. Show that X is a good integrator if and only if the set

$$\mathfrak{X}_{(1)} := \{H \bullet X_T : H \in \mathrm{b}\mathcal{E}, \|H\|_\infty \leqslant 1\}$$

is bounded in L^0, in the sense that $\lim_{n \to \infty} \sup_{Y \in \mathfrak{X}_{(1)}} P[|Y| \geqslant n] = 0$.

Recall that X is a good integrator if whenever $H^n, H \in \mathrm{b}\mathcal{E}$ with $H^n \to H$ uniformly in (ω, t), we have $H^n \bullet X_T \to H \bullet X_T$ in L^0.

Exercise 5.3 *(The spaces $\mathbb{L}$ and $\mathbb{D}$)* Fix a finite time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ is assumed to be complete. Let $\mathbb{L}$ and $\mathbb{D}$ denote the spaces of adapted LCRL and adapted RCLL processes, respectively. Define the metric

$$d(X^1, X^2) := E[1 \wedge (X^1 - X^2)^*_T] := E\Big[1 \wedge \sup_{0 \leqslant s \leqslant T} |X^1_s - X^2_s|\Big]$$

on both $\mathbb{L}$ and $\mathbb{D}$ (note that convergence with respect to d is exactly uniform (in t) convergence in probability). Show that when equipped with d, both $\mathbb{L}$ and $\mathbb{D}$ are complete metric spaces.

Hint: You may use that the space of (deterministic) LCRL (respectively RCLL) functionals on $[0, T]$ equipped with the supremum norm is a Banach space.

Exercise 5.4 *(Stopped good integrator)* Show that a stopped good integrator is a good integrator. That is, if $X = (X_t)_{0 \leqslant t \leqslant T}$ is a good integrator and τ is a stopping time, show that X^τ is a good integrator.

Exercise 5.5 *(Corollary 3.8)* Let $\mathcal{M}_{0,\mathrm{loc}}$ denote the space of local martingales null at zero. Show that if $M \in \mathcal{M}_{0,\mathrm{loc}}$ then $[M]^{1/2}$ is locally integrable.

Exercise 5.6 *(Approximation)* Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ is right-continuous, and let S be a semimartingale. Show in detail that for every $H \in \mathbb{L}$ such that $H_0 = 0$ almost surely, one can find a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathrm{b}\mathcal{E}$ with $d'_E(H^n \bullet S, H \bullet S) \to 0$ as $n \to \infty$.