Mathematical Finance

Exercise Sheet 6

Submit by 12:00 on Wednesday, November 6 via the course homepage.

Exercise 6.1 *(Bounded in L^0)* Show that a nonempty set $C \subseteq L^0$ is bounded in L^0 if and only if for every sequence $(X_n)_{n\in\mathbb{N}} \subseteq C$ and every sequence of scalars $\lambda_n \to 0$, we have $\lambda_n X_n \to 0$ in L^0.

Exercise 6.2 *(Quadratic covariation)* Recall that for a semimartingale S, the *optional quadratic variation* process is given by

$$[S] := S^2 - S_0^2 - 2\int S_- \,\mathrm{d}S.$$

For two semimartingales X and Y, we define the *optional quadratic covariation* process to be

$$[X,Y] := \frac{1}{4}([X+Y] - [X-Y]).$$

Note that this definition is "consistent" with the optional quadratic variation in the sense that $[X,X] = [X]$.

(a) Establish the integration by parts formula

$$XY = X_0Y_0 + \int X_- \,\mathrm{d}Y + \int Y_- \,\mathrm{d}X + [X,Y].$$

(b) Show that $\Delta[X,Y] = \Delta X \Delta Y$.

(c) Show that $\sum_{0<t\leqslant T}(\Delta X_t)^2 \leqslant [X]_T$.

In particular, $\sum_{0<t\leqslant T}(\Delta X_t)^2$ is P-a.s. convergent (while $\sum_{0<t\leqslant T}|\Delta X_t|$ need not converge).

Exercise 6.3 *(Semimartingales)* Show that $X \in \mathbb{D}$ is a semimartingale if and only if $d'_E(\lambda_n X, 0) \to 0$ whenever $\lambda_n \to 0$ in $\mathbb{R}$.