Coordinator
Chiara Rossato

# Mathematical Finance

## Exercise Sheet 9

*Submit by 12:00 on Wednesday, November 27 via the course homepage.*

**Exercise 9.1** *(Coherent risk measure)* Recall the map $\pi^s : L^\infty \to \mathbb{R}$ defined by

$$\pi^s(H) := \inf\left\{ v_0 \in \mathbb{R} : v_0 + \int_0^T \vartheta_u \, \mathrm{d}S_u \geqslant H \ P\text{-a.s. for some } \vartheta \in \Theta_{\mathrm{adm}} \right\}.$$

Prove that $\rho := -\pi^s$ is a *coherent risk measure*. That is, for all $H, H' \in L^\infty$,

1. $\pi^s(H) \leqslant \pi^s(H')$ if $H \leqslant H'$ $P$-a.s. *(monotonicity)*,
2. $\pi^s(H + c) = \pi^s(H) + c$ for all $c \in \mathbb{R}$ *(cash invariance)*,
3. $\pi^s(\lambda H) = \lambda \pi^s(H)$ for all $\lambda > 0$ *(positive homogeneity)*,
4. $\pi^s(H + H') \leqslant \pi^s(H) + \pi^s(H')$ *(subadditivity)*.

Deduce that $\pi^s$ is convex.

What happens in 3. for $\lambda = 0$?

**Exercise 9.2** *(Minimum principle)* Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $X = (X_t)_{t \geqslant 0}$ be a nonnegative RCLL supermartingale. Define the stopping time $\tau_0$ by

$$\tau_0 := \inf\{t \geqslant 0 : X_t \wedge X_{t-} = 0\}.$$

Show that $X \equiv 0$ on $[\![\tau_0, \infty[\![$ $P$-a.s.

*This result is known as the minimum principle for nonnegative supermartingales.*

**Exercise 9.3** *($\sigma$-martingales)*

(a) Let $Y = (Y_t)_{0 \leqslant t \leqslant T}$ be a RCLL process and $Q \approx P$ an equivalent measure with density process $Z$ given by $Z_t := \frac{dQ}{dP}|_{\mathcal{F}_t}$. Then $Y$ is a $Q$-$\sigma$-martingale if and only if $ZY$ is a $P$-$\sigma$-martingale.

*Hint. You may use Bayes theorem and the fact that the sum of two $\sigma$-martingales is a $\sigma$-martingale.*

(b) Show that if $S$ admits a $P$-equivalent $\sigma$-martingale density and $Q \approx P$ on $\mathcal{F}_T$, then $S$ also admits a $Q$-equivalent $\sigma$-martingale density.

**Exercise 9.4** *(A property of $\mathcal{Z}$)* Fix $Q \in \mathbb{P}_{e,\sigma}(S)$. Recall that for each $t \in [0,T]$, we let $\mathcal{Z}_t$ denote the space of RCLL martingales $Z$ such that $Z_s = \frac{\mathrm{d}R}{\mathrm{d}Q}|_{\mathcal{F}_s}$ for all $0 \leqslant s \leqslant T$ for some $R \in \mathbb{P}_{\mathrm{e},\sigma}(S)$ with $R = Q$ on $\mathcal{F}_t$.

Prove that if $Z^1, Z^2 \in \mathcal{Z}_t$ and $A \in \mathcal{F}_t$, then $Z^1 \mathbf{1}_A + Z^2 \mathbf{1}_{A^c} \in \mathcal{Z}_t$.