Mathematical Finance

Exercise Sheet 1

Submit by 12:00 on Wednesday, October 2 via the course homepage.

Exercise 1.1 *(Path regularity and measurability)* Let $S = (S_t)_{t \geqslant 0}$ be a real-valued stochastic process. Define the processes S^* and A by $S^*_t := \sup_{0 \leqslant r \leqslant t} S_r$ and $A_t := \int_0^t S_r \, \mathrm{d}r$ (when it exists), respectively.

(a) Show that if S is RCLL, then S^* is RCLL and A is well defined and continuous.

Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions.

(b) Show that if S is RCLL and adapted, then also S^* and A are adapted.

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and define the process $\vartheta = (\vartheta_t)_{t \geqslant 0}$ by $\vartheta_t := f(S_t, S^*_t, A_t)$.

Show that if S is adapted and continuous, then ϑ is predictable.

Solution 1.1

(a) Throughout this part, a single $\omega \in \Omega$ is fixed.

We first show that S^* is RCLL. Fix some $t_0 \in [0, \infty)$. Since S^* is nondecreasing by definition, the left and right limits $\lim_{t \uparrow t_0} S^*_t$ and $\lim_{t \downarrow t_0} S^*_t$ exist, and $\lim_{t \downarrow t_0} S^*_t \geqslant S^*_{t_0}$. It remains to show that $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0}$. To this end, fix $\varepsilon > 0$ and note that by right-continuity of S, there exists $\delta > 0$ such that $|S_t - S_{t_0}| < \varepsilon$ for all $t \in [t_0, t_0 + \delta]$. It follows that for all $t \in [t_0, t_0 + \delta]$, we have

$$
\begin{aligned}
S^*_t \leqslant S^*_{t_0+\delta} = \sup_{0 \leqslant r \leqslant t_0+\delta} S_r &= \max \left\{ \sup_{0 \leqslant r \leqslant t_0} S_r, \sup_{t_0 \leqslant r \leqslant t_0+\delta} S_r \right\} \\
&\leqslant \max \left\{ \sup_{0 \leqslant r \leqslant t_0} S_r, S_{t_0} + \varepsilon \right\} \leqslant \sup_{0 \leqslant r \leqslant t_0} S_r + \varepsilon = S^*_{t_0} + \varepsilon.
\end{aligned}
$$

Hence $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0} + \varepsilon$, and letting $\varepsilon \downarrow 0$ gives $\lim_{t \downarrow t_0} S^*_t \leqslant S^*_{t_0}$, as required.

For A, note first that any RCLL function has at most countably many discontinuities on any compact interval, and these form a null set for Lebesgue measure. So it is enough to integrate S only over those $r \in [0, t]$ where it is continuous, and then clearly A_t is well defined.

It remains to show that A is continuous. To this end, fix $0 \leqslant s < t < \infty$, and note that

$$|A_t - A_s| = \left| \int_s^t S_r \, \mathrm{d}r \right| \leqslant \int_s^t |S_r| \, \mathrm{d}r \leqslant |t-s| \sup_{s \leqslant r \leqslant t} |S_r|.$$

Since RCLL functions are locally bounded, $\sup_{s\leqslant r\leqslant t} |S_r|$ exists and is finite. It follows that A is continuous.

(b) We first show that S^* is adapted. Fix $t \geqslant 0$. Since S is right-continuous,

$$S^*_t = \sup_{0\leqslant r\leqslant t} S_r = \sup_{0\leqslant r\leqslant t,\, r\in\mathbb{Q}\cup\{t\}} |S_r|.$$

Using that S is adapted, it follows immediately that S^*_t is $\mathcal{F}_t$-measurable, and thus S^* is adapted.

It remains to show that A is adapted. To this end, consider for each $n \in \mathbb{N}$ the process $S^{(n)}$ defined by

$$S^{(n)}_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{\frac{k-1}{n} < t \leqslant \frac{k}{n}\}} S_{\frac{k}{n}}.$$

By (a.s.) right-continuity of S, we have $S^{(n)}_t \to S_t$ as $n \to \infty$ (a.s.). Since RCLL functions are bounded on compact intervals (here $[0,t]$), we can use the dominated convergence theorem to get

$$A_t = \int_0^t S_r \, \mathrm{d}r = \lim_{n\to\infty} \int_0^t S^{(n)}_r \, \mathrm{d}r = \lim_{n\to\infty} \sum_{k=1}^{\lceil nt\rceil} \tfrac{1}{n} S_{\frac{k}{n}} = \lim_{n\to\infty} \sum_{k=1}^{\lceil nt\rceil -1} \tfrac{1}{n} S_{\frac{k}{n}},$$

where the above limit is understood to hold almost surely. Since each sum $\sum_{k=1}^{\lceil nt\rceil-1} \frac{1}{n} S_{\frac{k}{n}}$ is $\mathcal{F}_t$-measurable, so is the limit $\lim_{n\to\infty} \sum_{k=1}^{\lceil nt\rceil-1} \frac{1}{n} S_{\frac{k}{n}}$. But A_t is equal to this limit up to a null set; so by completeness of $\mathbb{F}$, A_t is also $\mathcal{F}_t$-measurable, and hence A is adapted, as required.

(c) Since the processes S, S^*, A are adapted, so is ϑ.

Assume first that S is continuous. By repeating the argument in part (a), we can show that S^* is also continuous. We also know from part (a) that A is continuous, and thus so is ϑ. It now follows that ϑ is predictable, since it is an adapted and continuous process.

Exercise 1.2 *(Reparametrisation)* Fix a finite time horizon $T > 0$ and let $S = (S_t)_{0\leqslant t\leqslant T}$ be a semimartingale. Prove that there is a bijection between self-financing strategies $\varphi = (\varphi^0, \vartheta)$ and pairs

$$(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}.$$

Give explicitly the bijection map and its inverse.

Solution 1.2 Recall that a self-financing strategy is a pair $\varphi = (\varphi^0, \vartheta)$, where φ^0 is an adapted real-valued process and ϑ is a predictable, $\mathbb{R}^d$-valued and S-integrable process such that

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_r \,\mathrm{d}S_r, \qquad \forall t \in [0, T].$$

Now consider the map f defined on the family of self-financing strategies that sends each $\varphi = (\varphi^0, \vartheta)$ to (v_0, ϑ), where

$$v_0 = V_0(\varphi) = \varphi_0^0 + \vartheta_0^{\mathrm{tr}} S_0.$$

It is clear that $v_0 \in L^0(\mathcal{F}_0)$, and thus f is a well-defined map on the set of self-financing strategies into the space $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$. We show that f is a bijection.

Now consider the map g (which we show below to be the inverse of f) defined on the space $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$ that sends each (v_0, ϑ) to (φ^0, ϑ), where the process φ^0 is defined by

$$\varphi^0 = v_0 + \int \vartheta \,\mathrm{d}S - \vartheta^{\mathrm{tr}} S.$$

It is immediate that φ^0 is a real-valued adapted process. Moreover,

$$\begin{aligned} V_t(\varphi^0, \vartheta) - V_0(\varphi^0, \vartheta) &= \varphi_t^0 + \vartheta_t^{\mathrm{tr}} S_t - \varphi_0^0 - \vartheta_0^{\mathrm{tr}} S_0 \\ &= v_0 + \int_0^t \vartheta_r \,\mathrm{d}S_r - \vartheta_t^{\mathrm{tr}} S_t + \vartheta_t^{\mathrm{tr}} S_t - v_0 + \vartheta_0^{\mathrm{tr}} S_0 - \vartheta_0^{\mathrm{tr}} S_0 \\ &= \int_0^t \vartheta_r \,\mathrm{d}S_r. \end{aligned}$$

It follows that g is a well-defined map into the set of self-financing strategies.

To see that f is a bijection onto the set $L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$, it suffices to show that f and g are inverses of each other. To this end, take a self-financing strategy $\varphi = (\varphi^0, \vartheta)$, and compute

$$g \circ f(\varphi) = g(\varphi_0^0 + \vartheta_0^{\mathrm{tr}} S_0, \vartheta) = (\varphi_0^0 + \vartheta_0^{\mathrm{tr}} S_0 + \int \vartheta \,\mathrm{d}S - \vartheta^{\mathrm{tr}} S, \vartheta) = (\varphi^0, \vartheta) = \varphi,$$

where in the last step we used that φ is self-financing so that

$$\int \vartheta \,\mathrm{d}S = V(\varphi) - V_0(\varphi).$$

We have thus shown that $g \circ f$ is the identity map.

Similarly, take $(v_0, \vartheta) \in L^0(\mathcal{F}_0) \times \{\text{predictable } S\text{-integrable processes}\}$ and compute

$$f \circ g(v_0, \vartheta) = f\left(v_0 + \int \vartheta \, \mathrm{d}S - \vartheta^{\mathrm{tr}} S, \vartheta\right) = (v_0 - \vartheta_0^{\mathrm{tr}} S_0 + \vartheta_0^{\mathrm{tr}} S_0, \vartheta)$$
$$= (v_0, \vartheta).$$

Hence, $f \circ g$ is also the identity map, and this completes the proof.

Exercise 1.3 *(Brownian motion)* Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ be a filtered probability space satisfying the usual conditions, and let $B = (B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three-dimensional Brownian motion starting at $(1, 0, 0)$. Define the process $X = (X_t)_{t\geq 0}$ by $X_t := \|B_t\|$, where $\|\cdot\|$ denotes the Euclidean norm.

(a) Let $a, b > 0$ such that $a < 1 < b$, and consider the stopping time

$$\tau_{a,b} := \inf\{t \geq 0 : \; X_t \leq a \text{ or } X_t \geq b\}.$$

Show that $Y = (Y_t)_{t\geq 0}$ defined by $Y_t := (X_{\tau_{a,b}\wedge t})^{-1}$ is a bounded martingale.

(b) Show that $P\Big[X_t = 0 \text{ for some } t \geq 0\Big] = 0$.

(c) Show that X satisfies the SDE

$$\mathrm{d}X_t = \frac{1}{X_t}\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 1, \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a one-dimensional Brownian motion starting at 0.

(d) Let $(S, 1) = (S_t, 1)_{t\in[0,1]}$ be a continuous-time model with time horizon $T = 1$, where S satisfies the SDE

$$\mathrm{d}S_t = S_t\left(\left(\frac{1}{X_t} + 2\right)\mathrm{d}t + \mathrm{d}W_t\right), \quad S_0 = 1,$$

and where W is the Brownian motion introduced in (1). Show that S fails the NA condition by showing that the strategy $\theta = (\theta_t)_{t\in[0,1]}$, defined by $\theta_t := (S_t)^{-1}$, is an arbitrage opportunity.

Solution 1.3

(a) Consider the function $f(b) := g(\|b\|) := \|b\|^{-1}$. Then, for any $b \neq (0, 0, 0)$, we have

$$\Delta f(b) = g''(\|b\|) + \frac{2}{\|b\|} g'(\|b\|) = 0.$$

By applying Itô's formula, we see that for all $t \geq 0$,

$$Y_t = \frac{1}{X_{\tau_{a,b} \wedge t}} = f(B_{\tau_{a,b} \wedge t}) = 1 + \int_0^{\tau_{a,b} \wedge t} \nabla f(B_r) \bullet \mathrm{d}B_r \quad P\text{-a.s.},$$

which shows that Y is a local martingale. Moreover, since

$$0 \leq Y_t = \frac{1}{X_{\tau_{a,b} \wedge t}} \leq \frac{1}{a},$$

Y is uniformly bounded. Therefore, we conclude that Y is a true martingale.

(b) Since Y is a bounded martingale, we have

$$1 = \mathbb{E}\left[\lim_{t \to \infty} Y_t\right] = \mathbb{E}\left[(X_{\tau_{a,b}})^{-1}\right] = a^{-1} P[X_{\tau_{a,b}} = a] + b^{-1} P[X_{\tau_{a,b}} = b].$$

Additionally, since $P[X_{\tau_{a,b}} = a] + P[X_{\tau_{a,b}} = b] = 1$, we obtain that

$$P[X_{\tau_{a,b}} = a] = \frac{1 - b^{-1}}{a^{-1} - b^{-1}} \quad \text{and} \quad P[X_{\tau_{a,b}} = b] = \frac{a^{-1} - 1}{a^{-1} - b^{-1}}. \tag{2}$$

For any $c \geq 0$, we consider the stopping times

$$\tau_c := \inf\{t \geq 0 : X_t = c\}, \quad \text{and} \quad \sigma_c := \inf\{t \geq 0 : X_t \geq c\}.$$

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence decreasing to 0 such that $a_n < 1$ for all $n \in \mathbb{N}$. From (2), we deduce that for any fixed $b > 1$,

$$P[\tau_0 < \sigma_b] = P\left[\bigcap_{n \in \mathbb{N}} \{\tau_{a_n} < \sigma_b\}\right] = \lim_{n \to \infty} P[\tau_{a_n} < \sigma_b] = \lim_{n \to \infty} P[X_{\tau_{a_n,b}} = a_n] = 0.$$

Now, let $(b_n)_{n \in \mathbb{N}}$ be a sequence increasing to ∞ such that $b_n > 1$ for all $n \in \mathbb{N}$. Then, we deduce that

$$P[X_t = 0 \text{ for some } t \geq 0] = P\left[\bigcup_{n \in \mathbb{N}} \{\tau_0 < \sigma_{b_n}\}\right] = \lim_{n \to \infty} P[\tau_0 < \sigma_{b_n}] = 0.$$

(c) Since X is strictly positive P-a.s., we can apply Itô's formula to obtain

$$\mathrm{d}X_t = \frac{1}{\|X_t\|}\mathrm{d}t + \frac{B_t}{\|B_t\|} \bullet \mathrm{d}B_t \quad P\text{-a.s.} \tag{3}$$

Now, consider the process $W = (W_t)_{t \in [0,1]}$ defined by

$$W_t := \int_0^t \frac{B_r}{\|B_r\|} \bullet \mathrm{d}B_r. \tag{4}$$

W is a continuous local martingale with quadratic variation given by

$$\langle W \rangle_t = \int_0^t \left(\sum_{i=1}^3 \frac{(B_r^i)^2}{\|B_r\|^2}\right) \mathrm{d}r = \int_0^t \mathrm{d}r = t,$$

and hence a Brownian motion by Lévy's characterisation theorem.

(d) Using the result from part (c), the cumulative gains process $G(\theta) = (G_t(\theta))_{t\in[0,1]}$ satisfies the SDE

$$\mathrm{d}G_t(\theta) = \left(\frac{1}{X_t} + 2\right)\mathrm{d}t + \mathrm{d}W_t = \mathrm{d}X_t + 2\,\mathrm{d}t, \quad G_0 = 0. \tag{5}$$

Therefore, $G_t = X_t + 2t - X_0 = X_t + 2t - 1$ for any $t \in [0,1]$. Since X is strictly positive, this implies that the strategy θ is 1-admissible and an arbitrage opportunity.

Exercise 1.4 *(Geometric Brownian motion)* Fix constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion. Define the process $S = (S_t)_{t\geqslant 0}$ by

$$S_t := S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

The process $S = (S_t)_{t\geqslant 0}$ is called a *geometric Brownian motion* and is the stock price process in the *Black–Scholes model.*

Find $\lim_{t\to\infty} S_t$ (if it exists) for all possible parameter constellations.

Hint: You may use the law of the iterated logarithm.

Solution 1.4 We can rewrite S_t as

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{2t\log\log t}\frac{W_t}{\sqrt{2t\log\log t}}\right) \\ &= S_0 \exp\left(\sqrt{2t\log\log t}\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{t}{\sqrt{2t\log\log t}} + \sigma\frac{W_t}{\sqrt{2t\log\log t}}\right)\right). \end{aligned}$$

Since

$$\lim_{t\to\infty} \sqrt{2t\log\log t} = \infty \qquad \text{and} \qquad \lim_{t\to\infty} \frac{t}{\sqrt{2t\log\log t}} = \infty$$

and by the law of the iterated logarithm, it follows that

- when $\mu > \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = \infty;$$
- when $\mu < \frac{\sigma^2}{2}$,
$$\lim_{t\to\infty} S_t = 0;$$
- when $\mu = \frac{\sigma^2}{2}$,
$$\liminf_{t\to\infty} S_t = 0 \qquad \text{and} \qquad \limsup_{t\to\infty} S_t = \infty,$$
and hence $\lim_{t\to\infty} S_t$ does not exist.

Alternative solution: We can write

$$\log \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

By the law of the iterated logarithm, W_t grows more slowly than t as $t \to \infty$. So for $\mu > \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to \infty$, hence $S_t \to \infty$, and for $\mu < \frac{\sigma^2}{2}$, $\log(S_t/S_0) \to -\infty$, hence $S_t \to 0$. For $\mu = \frac{\sigma^2}{2}$, $\log(S_t/S_0) = \sigma W_t$ has $\limsup_{t\to\infty} \sigma W_t = \infty$ and $\liminf_{t\to\infty} \sigma W_t = -\infty$, hence $\limsup_{t\to\infty} S_t = \infty$ and $\liminf_{t\to\infty} S_t = 0$, so that $\lim_{t\to\infty} S_t$ does not exist.