Coordinator
Chiara Rossato

Mathematical Finance

Exercise Sheet 10

Submit by 12:00 on Wednesday, December 4 via the course homepage.

Exercise 10.1 *(Construction of ζ)* Let $S = (S_t)_{0 \leqslant t \leqslant T}$ be an RCLL process with $S_0 = 0$.

(a) Assume S is locally bounded, so that there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times increasing stationarily to T with S^{τ_n} bounded for each n. Show that there exists a strictly positive predictable process $\zeta \in L(S)$ such the random variable

$$(\zeta \bullet S)_T^* := \sup_{0 \leqslant t \leqslant T} |\zeta \bullet S_t|$$

is bounded.

(b) Assume instead that S is a σ-martingale. Show that there exists a strictly positive predictable process $\zeta \in L(S)$ such the $(\zeta \bullet S)_T^*$ is integrable.

Solution 10.1

(a) Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times increasing stationarily to T such that $|S^{\tau_n}| \leqslant b_n$ for each n, where $b_n < \infty$ is some constant. Define the process ζ by

$$\zeta := \mathbf{1}_{[\![0]\!]} + \sum_{n=1}^{\infty} \frac{1}{2^n (b_{n-1} + b_n)} \mathbf{1}_{]\!]\tau_{n-1}, \tau_n]\!]}.$$

As $\tau_n = T$ eventually with probability 1, it follows that for almost all ω, the above series is really a finite sum (for ω such that $\tau_n(\omega) < T$ for all n, redefine $\zeta(\omega) = 1$). We thus have that $\zeta \bullet S$ is well defined. Moreover, we have

$$|(\zeta \bullet S)_T^*| \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n (b_{n-1} + b_n)} |S^{\tau_n} - S^{\tau_{n-1}}| \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Since S is strictly positive and predictable by construction, this completes the proof.

(b) As S is a σ-martingale, there exist a local martingale M and a strictly positive integrand $\psi \in L(M)$ such that $S = \psi \bullet M$. As ψ is strictly positive, then $\frac{1}{\psi}$ is well-defined, and we have

$$\frac{1}{\psi} \bullet S = \frac{1}{\psi} \bullet (\psi \bullet M) = M.$$

By Exercise 3.1, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing stationarily to T such that for each n, $M^{\tau_n} \in \mathcal{H}^1$, i.e. $(M^{\tau_n})^*_T \in L^1$. Define the process ζ by

$$\zeta := \mathbf{1}_{[\![0]\!]} + \sum_{n=1}^{\infty} \frac{1}{2^n(\|(M^{\tau_n})^*_T\|_{L^1} + \|(M^{\tau_{n-1}})^*_T\|_{L^1})} \frac{1}{\psi} \mathbf{1}_{]\!]\tau_{n-1},\tau_n]\!]}.$$

As $\tau_n = T$ eventually with probability 1, it follows that for almost all ω, the above series is really a finite sum (for ω such that $\tau_n(\omega) < T$ for all n, redefine $\zeta(\omega) = 1$). We thus have that $\zeta \bullet S$ is well defined. Moreover, we have

$$\begin{aligned}\|(\zeta \bullet S)^*_T\|_{L^1} &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n(\|(M^{\tau_n})^*_T\|_{L^1} + \|(M^{\tau_{n-1}})^*_T\|_{L^1})} \|M^{\tau_n} - M^{\tau_{n-1}}\|_{L^1} \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.\end{aligned}$$

Since S is strictly positive and predictable by construction, this completes the proof.

Exercise 10.2 *(Sum of σ-martingales is a σ-martingale)* Let S^1 and S^2 be σ-martingales. Show that the sum $S^1 + S^2$ is again a σ-martingale.

Solution 10.2 There exist local martingales M^1 and M^2 and strictly positive integrands $\psi^1 \in L(M^1)$ and $\psi^2 \in L(M^2)$ such that $S^1 - S^1_0 = \psi^1 \bullet M^1$ and $S^2 - S^2_0 = \psi^2 \bullet M^2$. Now set $\varphi^1 := \frac{1}{\psi^1}$ and $\varphi^2 := \frac{1}{\psi^2}$, which are well defined and strictly positive since ψ^1 and ψ^2 are. Note that the integral process

$$\varphi^1 \bullet S^1 = \varphi^1 \bullet (\psi^1 \bullet M^1) = (\varphi^1\psi^1) \bullet M^1 = M^1$$

is a local martingale, and similarly $\varphi^2 \bullet S^2 = M^2$ is a local martingale. Now define $\varphi := \varphi^1 \wedge \varphi^2$, which is strictly positive since φ^1 and φ^2 are. We have

$$\varphi \bullet S^1 = \frac{\varphi}{\varphi^1} \bullet (\varphi^1 \bullet S^1) \quad \text{and} \quad \varphi \bullet S^2 = \frac{\varphi}{\varphi^2} \bullet (\varphi^2 \bullet S^2),$$

and since $\frac{\varphi}{\varphi^1}$ and $\frac{\varphi}{\varphi^2}$ are bounded (by 1), we get that $\varphi \bullet S^1$ and $\varphi \bullet S^2$ are local martingales. Since the sum of local martingales is a local martingale, we get that the integral process

$$\varphi \bullet (S^1 + S^2) = \varphi \bullet S^1 + \varphi \bullet S^2$$

is a local martingale, and thus

$$S^1 + S^2 = S^1_0 + S^2_0 + \frac{1}{\varphi} \bullet \Big(\varphi \bullet (S^1 + S^2)\Big)$$

is a σ-martingale, as required.

Exercise 10.3 *(Density of $\mathbb{P}_{\mathrm{e},\sigma}$ in $\mathbb{P}_{\mathrm{a},\sigma}$)* Let $S = (S_t)_{0\leqslant t\leqslant T}$ be a P-semimartingale. Recall the set $\mathbb{P}_{\mathrm{a},\sigma}(S)$ defined by

$$\mathbb{P}_{\mathrm{a},\sigma}(S) := \{Q \ll P \text{ on } \mathcal{F}_T : S \text{ is a } Q\text{-}\sigma\text{-martingale}\}.$$

(a) Show that the sets $\mathbb{P}_{\mathrm{a},\sigma}(S)$ and $\mathbb{P}_{\mathrm{e},\sigma}(S)$ are convex.

(b) Assume that $\mathbb{P}_{\mathrm{e},\sigma}(S) \neq \varnothing$. Show that $\mathbb{P}_{\mathrm{e},\sigma}(S)$ is $L^1(P)$-dense in $\mathbb{P}_{\mathrm{a},\sigma}(S)$, in the sense that for each measure $Q \in \mathbb{P}_{\mathrm{a},\sigma}(S)$, there is a sequence $(Q^n)_{n\in\mathbb{N}} \subseteq \mathbb{P}_{\mathrm{e},\sigma}(S)$ such that $Z^n \to Z$ in $L^1(P)$, where Z^n and Z denote the densities of Q^n and Q with respect to P, respectively.

Solution 10.3

(a) We first prove that $\mathbb{P}_{\mathrm{a},\sigma}(S)$ is convex. So take $P^1, P^2 \in \mathbb{P}_{\mathrm{a},\sigma}(S)$ and $\lambda \in (0,1)$. We need to show that $P^0 := \lambda P^1 + (1-\lambda)P^2 \in \mathbb{P}_{\mathrm{a},\sigma}(S)$. Fix $i \in \{1,2\}$. By the definition of $\mathbb{P}_{\mathrm{a},\sigma}(S)$, there exist a P^i-local martingale M^i and a strictly positive integrand $\psi^i \in L(M^i)$ such that $S - S_0 = \psi^i \bullet M^i$. As ψ^i is strictly positive, the process $\varphi^i := \frac{1}{\psi^i}$ is well defined. Moreover, we have

$$\varphi^i \bullet S = \varphi^i \bullet (\psi^i \bullet M^i) = (\varphi^i \psi^i) \bullet M^i = M^i.$$

Now, define $\varphi := \varphi^1 \wedge \varphi^2$. As φ^1 and φ^2 are predictable and strictly positive, so is φ. We show that $\varphi \bullet S$ is a P^0-local martingale, since then

$$S = S_0 + \frac{1}{\varphi} \bullet (\varphi \bullet S)$$

will be a P^0-σ-martingale. To this end, first note that

$$\varphi \bullet S = \frac{\varphi}{\varphi^i} \bullet (\varphi^i \bullet S) = \frac{\varphi}{\varphi^i} \bullet M^i.$$

Since M^i is a P^i-local martingale and $\frac{\varphi}{\varphi^i}$ is bounded (by 1), we know that $\varphi \bullet S$ is a P^i-local martingale. By Exercise 3.1, every local martingale is locally in $\mathcal{H}^1$. So let $(\tau_n^i)_{n\in\mathbb{N}}$ be a sequence of stopping times increasing stationarily to T such that $(\varphi \bullet S)^{\tau_n^i} \in \mathcal{H}^1(P^i)$ for each n. Define $\tau_n := \tau_n^1 \wedge \tau_n^2$. Then $(\tau_n)_{n\in\mathbb{N}}$ increases stationarily to T and $(\varphi \bullet S)^{\tau_n} \in \mathcal{H}^1(P^1) \cap \mathcal{H}^1(P^2)$. Now fix $0 \leqslant s \leqslant t \leqslant T$. We claim that

$$E_{P^0}[(\varphi \bullet S)_t^{\tau_n} \mid \mathcal{F}_s] = (\varphi \bullet S)_s^{\tau_n}.$$

To this end, take $A \in \mathcal{F}_s$, and using $E_{P^0}[\cdot] = \lambda E_{P^1}[\cdot] + (1-\lambda)E_{P^2}[\cdot]$, we can write

$$\begin{aligned} E_{P^0}[(\varphi \bullet S)_s^{\tau_n} \mathbf{1}_A] &= \lambda E_{P^1}[(\varphi \bullet S)_s^{\tau_n} \mathbf{1}_A] + (1-\lambda)E_{P^2}[(\varphi \bullet S)_s^{\tau_n} \mathbf{1}_A] \\ &= \lambda E_{P^1}[(\varphi \bullet S)_t^{\tau_n} \mathbf{1}_A] + (1-\lambda)E_{P^2}[(\varphi \bullet S)_t^{\tau_n} \mathbf{1}_A] \\ &= E_{P^0}[(\varphi \bullet S)_t^{\tau_n} \mathbf{1}_A]. \end{aligned}$$

It follows immediately that $E_{P^0}[(\varphi \bullet S)^{\tau_n}_t \mid \mathcal{F}_s] = (\varphi \bullet S)^{\tau_n}_s$, and thus $\varphi \bullet S$ is a P^0-local martingale, so that S is a P^0-σ-martingale.

Finally, take $A \in \mathcal{F}_T$ such that $P[A] = 0$. Then $P^1[A] = P^2[A] = 0$, and hence also $P^0[A] = 0$. We have thus shown that $P^0 \in \mathbb{P}_{\mathrm{a},\sigma}(S)$.

Now if we have that $P^1, P^2 \in \mathbb{P}_{\mathrm{e},\sigma}(S)$, the above gives that $P^0 \in \mathbb{P}_{\mathrm{a},\sigma}(S)$, and thus it remains to show that $P \ll P^0$ on $\mathcal{F}_T$. To this end, take $A \in \mathcal{F}_T$ such that $P^0[A] = 0$. As $\lambda \neq 0$, this means that $P^1[A] = 0$, and hence $P[A] = 0$ since $P \approx P^1$. This completes the proof.

(b) Fix $Q \in \mathbb{P}_{\mathrm{a},\sigma}(S)$. We need to find a sequence $(Q^n)_{n\in\mathbb{N}} \subseteq \mathbb{P}_{\mathrm{e},\sigma}(S)$ such that $Z^n \to Z$ in $L^1(P)$, where Z^n and Z denote the densities of Q^n and Q with respect to P, respectively. To this end, fix an arbitrary $Q^0 \in \mathbb{P}_{\mathrm{e},\sigma}(S)$ (which exists, since $\mathbb{P}_{\mathrm{e},\sigma}(S) \neq \varnothing$), and for each $n \in \mathbb{N}$, define $Q^n := \frac{1}{n}Q^0 + (1 - \frac{1}{n})Q$. As $Q^0, Q \in \mathbb{P}_{\mathrm{a},\sigma}(S)$, we know by part (a) that also $Q^n \in \mathbb{P}_{\mathrm{a},\sigma}(S)$. To see that $Q^n \in \mathbb{P}_{\mathrm{e},\sigma}(S)$, it thus suffices to show that $P \ll Q^n$ on $\mathcal{F}_T$. So take $A \in \mathcal{F}_T$ with $Q^n[A] = 0$. Then $Q^0[A] = 0$, and since $Q^0 \approx P$, we also get $P[A] = 0$ as required. So indeed $(Q^n)_{n\in\mathbb{N}} \subseteq \mathbb{P}_{\mathrm{e},\sigma}(S)$. Now let Z^n, Z^0 and Z denote the densities of Q^n, Q^0 and Q with respect to P, respectively. We have $Z^n = \frac{1}{n}Z^0 + (1 - \frac{1}{n})Z$, and thus

$$\|Z^n - Z\|_{L^1(P)} = \frac{1}{n}\|Z^0 - Z\|_{L^1(P)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

This completes the proof.