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Mathematical Finance

Exercise Sheet 12

Submit by 12:00 on Wednesday, December 18 via the course homepage.

Exercise 12.1 *(Some properties of u)* Let $U : (0, \infty) \to \mathbb{R}$ be a concave and increasing function. Define the function $u : (0, \infty) \to (-\infty, +\infty]$ by

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)],$$

where $\mathcal{V}(x) := \{x + G(\vartheta) : \vartheta \in \Theta^x_{\text{adm}}\}$.

(a) Show that u is concave and increasing.

(b) If additionally $u(x_0) < \infty$ for some $x_0 > 0$, show that $u(x) < \infty$ for all $x > 0$.

Solution 12.1

(a) We first prove that u is concave. Let $x, y \in (0, \infty)$ and $\lambda \in (0, 1)$ be fixed. We need to show that

$$u\Big(\lambda x + (1-\lambda)y\Big) \geqslant \lambda u(x) + (1-\lambda)u(y).$$

First note that if either $u(x)$ or $u(y)$ is $-\infty$, then the inequality holds trivially. So assume that $u(x), u(y) > -\infty$. Take $x + G(\vartheta^x) \in \mathcal{V}(x)$ and $y + G(\vartheta^y) \in \mathcal{V}(y)$ with $U(x + G(\vartheta^x))^-$ and $U(y + G(\vartheta^y))^-$ both in L^1. Then

$$\lambda\Big(x + G(\vartheta^x)\Big) + (1-\lambda)\Big(y + G(\vartheta^y)\Big) = \lambda x + (1-\lambda)y + G\Big(\lambda\vartheta^x + (1-\lambda)\vartheta^y\Big).$$

As U is concave, we have

$$\lambda U\Big(x+G(\vartheta^x)\Big)+(1-\lambda)U\Big(y+G(\vartheta^y)\Big) \leqslant U\Big(\lambda x+(1-\lambda)y+G\Big(\lambda\vartheta^x+(1-\lambda)\vartheta^y\Big)\Big).$$

So also $U(\lambda x + (1-\lambda)y + G\lambda\vartheta^x + (1-\lambda)\vartheta^y))^- \in L^1$. Furthermore, since $\lambda\vartheta^x + (1-\lambda)\vartheta^y \in \Theta^{\lambda x+(1-\lambda)y}_{\text{adm}}$, we can take expectations, which yields

$$\lambda E\Big[U\Big(x + G(\vartheta^x)\Big)\Big] + (1-\lambda)E\Big[U\Big(y + G(\vartheta^y)\Big)\Big] \leqslant u\Big(\lambda x + (1-\lambda)y\Big).$$

Finally, taking the supremum over all $x + G(\vartheta^x) \in \mathcal{V}(x)$ and $y + G(\vartheta^y) \in \mathcal{V}(y)$ with integrable negative parts gives the required inequality.

It remains to prove that u is increasing. This follows from the fact that $\Theta^x_{\mathrm{adm}} \subseteq \Theta^y_{\mathrm{adm}}$ for $0 < x < y$. Indeed, for $x + G(\vartheta^x) \in \mathcal{V}(x)$ so that $\vartheta^x \in \Theta^x_{\mathrm{adm}}$, we have $y + G(\vartheta^x) \in \mathcal{V}(y)$, and as U is increasing, this implies

$$E\Big[U\Big(x + G(\vartheta^x)\Big)\Big] \leqslant E\Big[U\Big(y + G(\vartheta^x)\Big)\Big] \leqslant u(y).$$

Taking the supremum over all $\vartheta^x \in \Theta^x_{\mathrm{adm}}$ gives $u(x) \leqslant u(y)$, completing the proof.

(b) As u is increasing, we know that $u(x) < \infty$ for all $x < x_0$. It thus remains to show that $u(x) < \infty$ for all $x > x_0$. By choosing $\lambda \in (0,1)$ small enough, we can find $y \in (0, x_0)$ such that

$$x_0 = \lambda x + (1-\lambda)y.$$

By concavity of u, we have

$$\lambda u(x) + (1-\lambda)u(y) \leqslant u(x_0) < \infty,$$

which gives the result because $u(y) \leqslant u(x_0) < \infty$ and $u(y) \geqslant U(y) > -\infty$.

Exercise 12.2 *(Utility in a market with arbitrage)* Consider a general market with finite time horizon T. Let $U : (0,\infty) \to \mathbb{R}$ be an increasing and concave utility function. Suppose that U is unbounded from above and that either the market admits a 0-admissible arbitrage opportunity, or we are in finite discrete time and the market admits an (admissible) arbitrage opportunity. Show that in both cases, we have $u \equiv \infty$.

Without imposing that U is unbounded from above, what can you say about the relationship between $u(x)$ and $U(x)$ as $x \to \infty$?

Solution 12.2 By assumption, there exists $\vartheta \in \Theta_{\mathrm{adm}}$ such that $G_T(\vartheta) \geqslant 0$ P-a.s. and $P[G_T(\vartheta) > 0] > 0$. By Exercise 4.2, we may assume that ϑ is 0-admissible, and so also $n\vartheta$ is 0-admissible for each $n \in \mathbb{N}$. It follows that $x + nG_T(\vartheta) \in \mathcal{V}(x)$ for every $x > 0$ and $n \in \mathbb{N}$. So setting $A := \{G_T(\vartheta) > 0\}$, we have that for all $x > 0$ and $n \in \mathbb{N}$,

$$u(x) \geqslant E\Big[U\Big(x + nG_T(\vartheta)\Big)\Big] = E\Big[U\Big(x + nG_T(\vartheta)\Big)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big].$$

As U is increasing, we can let $n \to \infty$ and apply the monotone convergence theorem to get that for all $x > 0$,

$$u(x) \geqslant E\Big[U(\infty)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big].$$

Note that U is increasing gives that the limit $U(\infty) := \lim_{x\to\infty} U(x) \in \mathbb{R} \cup \{\infty\}$ exists. Since U is unbounded from above we have $U(\infty) = \infty$, and as $P[A] > 0$, we can conclude that $u \equiv \infty$, as required.

Now suppose that U is not necessarily unbounded from above. We still have

$$u(x) \geqslant E\Big[U(\infty)\mathbf{1}_A\Big] + E\Big[U(x)\mathbf{1}_{A^c}\Big] = U(\infty)P[A] + U(x)P[A^c].$$

Also, by the definition of u, $u(x) \leqslant U(\infty)$ as U is increasing. So for each $x > 0$,

$$U(\infty)P[A] + U(x)P[A^c] \leqslant u(x) \leqslant U(\infty).$$

Letting $x \to \infty$ in the above gives $u(\infty) = U(\infty)$. This completes the problem.

Exercise 12.3 *(Utility in a complete market)* Consider a financial market modelled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function such that $u(x) < \infty$ for some (and hence for all) $x \in (0,\infty)$. Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Assume furthermore that $\mathcal{F}_0$ is trivial.

(a) Show that $h \leqslant z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. for all $h \in \mathcal{D}(z)$, and deduce that

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

(b) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j defined in the lecture notes is in $C^1((z_0,\infty);\mathbb{R})$ and satisfies

$$j'(z) = E\left[\frac{\mathrm{d}Q}{\mathrm{d}P}J'\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right], \quad z \in (z_0,\infty).$$

(c) Set $x_0 := \lim_{z\downarrow z_0}(-j'(z))$ and fix $x \in (0,x_0)$. Let $z_x \in (z_0,\infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I(z_x\frac{\mathrm{d}Q}{\mathrm{d}P})$ is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} E[U(f)].$$

Solution 12.3

For notational convenience, we denote by $Z^Q = (Z^Q_t)_{0\leqslant t\leqslant T}$ the density process of Q with respect to P, so that $Z^Q_T = \frac{\mathrm{d}Q}{\mathrm{d}P}$.

(a) Recall that in general, a payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if and only if the supremum

$$\sup_{Q^0\in\mathbb{P}_{\mathrm{e},\sigma}} E_{Q^0}[H]$$

is finite and attained at some $Q^* \in \mathbb{P}_{\mathrm{e},\sigma}$. In our setting, $\mathbb{P}_{\mathrm{e},\sigma}$ is the singleton set $\{Q\}$, so that a payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if and only if $E_Q[H] < \infty$, i.e. if and only if $H \in L^1_+(Q,\mathcal{F}_T)$.

Now we recall that

$$\mathcal{D}(z) := \{h \in L^0_+(\mathcal{F}_T) : \exists Z \in \mathcal{Z}(z) \text{ with } h \leqslant Z_T\}.$$

So take $h \in \mathcal{D}(z)$ and suppose for contradiction that we do not have $h \leqslant zZ^Q_T$ P-a.s. Then setting $A := \{h > zZ^Q_T\}$, we have $P[A] > 0$. Now define the process $M = (M_t)_{0\leqslant t\leqslant T}$ by

$$M_t := E_Q[\mathbf{1}_A \mid \mathcal{F}_t].$$

Then M is a nonnegative Q-martingale with $M_0 = Q[A] > 0$ because $Q \approx P$. Since $E_Q[M_T] \leqslant 1 < \infty$, it follows that $M_T \in L^0_+(\mathcal{F}_T)$ is attainable so that there exists some $\vartheta \in \Theta_{\text{adm}}$ with

$$M = M_0 + G(\vartheta).$$

Since M is nonnegative, we must have $\vartheta \in \Theta^{M_0}_{\text{adm}}$ and hence $M \in \mathcal{V}(M_0)$.

Now, since $h \in \mathcal{D}(z)$, there exists $Z \in \mathcal{Z}(z)$ such that $h \leqslant Z_T$. By the definition of $\mathcal{Z}(z)$, the product ZM is a P-supermartingale. We thus have

$$E[hM_T] \leqslant E[Z_T M_T] \leqslant E[Z_0 M_0] = zM_0.$$

Also, we have $E[zZ^Q_T M_T] = E_Q[zM_T] = zM_0$, and thus

$$E\left[\left(h - zZ^Q_T\right) M_T\right] \leqslant 0.$$

But recalling $M_T = \mathbf{1}_A$ and $P[A] > 0$ gives

$$E\left[\left(h - zZ^Q_T\right) M_T\right] > 0,$$

which gives a contradiction. Hence we must have $h \leqslant zZ^Q_T$ P-a.s., as required. In particular, as any $Z_T \in \mathcal{D}(z)$ for $Z \in \mathcal{Z}(z)$, this gives $Z_T \leqslant zZ^Q_T$ for any $Z \in \mathcal{Z}(z)$.

It remains to show $j(z) = E[J(zZ^Q_T)]$. First we recall that

$$j(z) := \inf_{Z\in\mathcal{Z}(z)} E[J(Z_T)].$$

For each $Z \in \mathcal{Z}(z)$ we have $Z_T \leqslant zZ^Q_T$. As J is decreasing, we have

$$J(Z_T) \geqslant J\left(zZ^Q_T\right),$$

and thus

$$E[J(Z_T)] \geqslant E\left[J\left(zZ^Q_T\right)\right].$$

Taking the infimum over all $Z \in \mathcal{Z}(z)$ gives

$$j(z) \geqslant E\left[J\left(zZ^Q_T\right)\right].$$

As $zZ^Q_T \in \mathcal{Z}(z)$, this concludes the proof.

(b) Note that $0 \leqslant z_0 < \infty$ by Theorem 12.4, and also by Theorem 12.4, we have that $j(z) < \infty$ for $z \in (z_0, \infty)$.

Now recall that J is in C^1 and strictly decreasing. We can thus define the function $g : (z_0, \infty) \to [-\infty, 0]$ by

$$g(s) := E[Z_T^Q J'(sZ_T^Q)].$$

Moreover, as J is also strictly convex, J' is increasing, and thus g is also increasing since $Z_T^Q > 0$. As g is negative-valued, it follows from the dominated convergence theorem that if $g(s_0) > -\infty$ for some $s_0 > z_0$, we have that g is continuous on (s_0, ∞).

Next, since $\frac{\mathrm{d}}{\mathrm{d}s} J(sZ_T^Q) = Z_T^Q J'(sZ_T^Q)$ by the chain rule, we have by the fundamental theorem of calculus that for $z_0 < z_1 < z_2 < \infty$,

$$J(z_2 Z_T^Q) - J(z_1 Z_T^Q) = \int_{z_1}^{z_2} Z_T^Q J'(sZ_T^Q) \, \mathrm{d}s.$$

By part (a), we know that $j(z) = E[J(zZ_T^Q)]$. Thus taking expectations of both sides in the above gives

$$j(z_2) - j(z_1) = E\left[\int_{z_1}^{z_2} Z_T^Q J'(sZ_T^Q) \, \mathrm{d}s\right] = \int_{z_1}^{z_2} E[Z_T^Q J'(sZ_T^Q)] \, \mathrm{d}s = \int_{z_1}^{z_2} g(s) \, \mathrm{d}s,$$

where the second step uses the Fubini–Tonelli theorem, keeping in mind that the integrand is strictly negative.

Note that by the definition of z_0, we have that $j(z_2) - j(z_1)$ is finite, and thus the function g is finite a.e. on (z_0, ∞). From the above, we can conclude that g is continuous and finite on (z_0, ∞). By dividing by $z_2 - z_1$ and letting $z_2 \to z_1$, we get that

$$j'(z) = E[Z_T^Q J'(zZ_T^Q)] = g(z)$$

as required. Now since g is continuous on (z_0, ∞), we have $j \in C^1((z_0, \infty); \mathbb{R})$, completing the proof.

(c) Before establishing that f^* is a solution to the primal problem, we first need to check that $f^* \in \mathcal{C}(x)$. To this end, recall that $f \in \mathcal{C}(x)$ if and only if

$$\sup_{h \in \mathcal{D}(1)} E[fh] \leqslant x.$$

By part (a), this is equivalent to

$$E[fZ_T^Q] \leqslant x.$$

Now by the definition of f^* and I, we have

$$E[f^* Z_T^Q] = E[I(z_x Z_T^Q) Z_T^Q] = E[-J'(z_x Z_T^Q) Z_T^Q].$$

Moreover, by part (b), we have $E[Z_T^Q J'(z_x Z_T^Q)] = j'(z_x)$, and since $-j'(z_x) = x$ by definition of z_x, we have

$$E[f^* Z_T^Q] = x$$

and thus in particular $f^* \in \mathcal{C}(x)$, as required.

Next, we establish that f^* is a solution to the primal problem. So fix $f \in \mathcal{C}(x)$. We need to show that $E[U(f^*)] \geqslant E[U(f)]$. We may thus assume without loss of generality that $E[U(f)] > -\infty$. Now since U is in C^1 and strictly concave on $(0, \infty)$, and since $f^* > 0$ P-a.s., we have

$$U(f) - U(f^*) \leqslant U'(f^*)(f - f^*),$$

with strict inequality on the event $\{f \neq f^*\}$. Now note that

$$U'(f^*) = U'\Big(I(z_x Z_T^Q)\Big) = z_x Z_T^Q.$$

Thus taking expectations of the above inequality yields

$$E\Big[U(f) - U(f^*)\Big] \leqslant E\Big[z_x Z_T^Q (f - f^*)\Big],$$

and since $E[Z_T^Q f^*] = x$ and $E[Z_T^Q f] \leqslant x$ and $z_x > 0$, we have

$$E[U(f) - U(f^*)] \leqslant 0,$$

and the inequality is strict when $P[f \neq f^*] > 0$. It follows immediately that f^* is the unique solution to the primal problem. This completes the proof.

Exercise 12.4 *(The Merton problem)* Consider the Black–Scholes market given by

$$\begin{aligned} d\tilde{S}_0^0 &= r\tilde{S}_t^0 \, dt, & \tilde{S}_0^0 &= 1, \\ d\tilde{S}_t^1 &= \tilde{S}_t^1(\mu \, dt + \sigma \, dW_t), & \tilde{S}_0^1 &= s > 0. \end{aligned}$$

Let $U : (0, \infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma} x^\gamma$, where $\gamma \in (-\infty, 1) \backslash \{0\}$. We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$).

(a) Show that for $z > 0$,

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \frac{(\mu - r)^2 T}{\sigma^2}\right).$$

(b) Show that the unique solution to the primal problem

$$u(x) = \sup_{f \in \mathcal{C}(x)} E[U(f)], \quad x \in (0, \infty),$$

is given by $f_x^* := x\mathcal{E}(\frac{1}{1-\gamma} \frac{\mu - r}{\sigma} R)_T$, where the process $R = (R_t)_{0 \leqslant t \leqslant T}$ is defined by $R_t = W_t + \frac{\mu - r}{\sigma} t$.

(c) Deduce that $f_x^* = V_T(x, \vartheta^x)$, where the integrand $\vartheta^x = (\vartheta_t^x)_{0 \leqslant t \leqslant T}$ is given by

$$\vartheta_t^x = \frac{x}{S_t^1} \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2} \mathcal{E}\left(\frac{1}{1-\gamma} \frac{\mu - r}{\sigma} R \right)_t, \quad x \in (0, \infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2} \frac{\gamma}{1-\gamma} \frac{(\mu - r)^2}{\sigma^2} T \right), \quad x \in (0, \infty).$$

(d) For any x-admissible ϑ with $V(x, \vartheta) > 0$, denote by

$$\pi_t := \frac{\vartheta_t S_t^1}{V_t(x, \vartheta)}$$

the fraction of wealth that is invested in the stock. Show that the optimal strategy ϑ^x is given by the *Merton proportion*

$$\pi_t^* = \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2}.$$

Solution 12.4

(a) In the Black–Scholes model, there exists a unique EMM Q, and thus Exercise 12.3(a) is applicable. We hence have

$$j(z) = E\left[J\left(z \frac{\mathrm{d}Q}{\mathrm{d}P} \right) \right].$$

To compute this, we start by writing

$$J(y) = \sup_{x>0} \Big(U(x) - xy \Big) = \sup_{x>0} \left(\frac{1}{\gamma} x^\gamma - xy \right).$$

Taking the derivative of $\frac{1}{\gamma} x^\gamma - xy$ with respect to x and setting it equal to zero, we get $x = y^{\frac{1}{\gamma-1}}$, and hence

$$J(y) = \frac{1}{\gamma} y^{\frac{\gamma}{\gamma-1}} - y^{\frac{\gamma}{\gamma-1}} = \frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}}.$$

We also recall that in the Black–Scholes model,

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \mathcal{E}(-\lambda W)_T,$$

where $\lambda := \frac{\mu - r}{\sigma}$. So we have

$$\begin{aligned} j(z) &= E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right] \\ &= \frac{1-\gamma}{\gamma} z^{\frac{\gamma}{\gamma-1}} E\left[\mathcal{E}(-\lambda W)_T^{\frac{\gamma}{\gamma-1}}\right] \\ &= \frac{1-\gamma}{\gamma} z^{\frac{\gamma}{\gamma-1}} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}W_T - \frac{1}{2}\frac{\lambda^2\gamma}{\gamma-1}T\right)\right] \\ &= \frac{1-\gamma}{\gamma} z^{\frac{\gamma}{\gamma-1}} \exp\left(-\frac{1}{2}\frac{\lambda^2\gamma}{\gamma-1}T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma}W\right)_T\right] \\ &= \frac{1-\gamma}{\gamma} z^{\frac{\gamma}{\gamma-1}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right), \end{aligned}$$

where in the last step we use that $\mathcal{E}(aW)$ is a P-martingale for each $a \in \mathbb{R}$. Substituting $\lambda = \frac{\mu-r}{\sigma}$ then gives the result.

(b) First, note that $j(z) < \infty$ for some $z \in (0,\infty)$ implies that

$$u(x) \leqslant j(z) + zx < \infty, \quad x \in (0,\infty).$$

We computed $J(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}}$, and hence $J'(z) = -z^{-\frac{1}{1-\gamma}}$. Now fix $x > 0$. With the same notation as in Exercise 12.3, we have

$$\begin{aligned} f_x^* &= -J'\left(z_x\frac{dQ}{dP}\right) = z_x^{-\frac{1}{1-\gamma}}\Big(\mathcal{E}(-\lambda W)_T\Big)^{-\frac{1}{1-\gamma}} \\ &= -j'(z_x)\exp\left(-\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right)\exp\left(\frac{\lambda}{1-\gamma}W_T + \frac{1}{2}\frac{\lambda^2}{1-\gamma}T\right) \\ &= x\exp\left(\frac{\lambda}{1-\gamma}(W_T+\lambda T) - \frac{1}{2}\frac{\lambda^2}{(1-\gamma)^2}T\right) \\ &= x\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_T. \end{aligned}$$

This completes the proof.

(c) Fix $x > 0$. By the definition of the stochastic exponential and using that $\lambda = \frac{\mu-r}{\sigma}$, we have

$$\begin{aligned} f_x^* &= x\left(1 + \int_0^T \mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma}\,\mathrm{d}R_t\right) \\ &= x + \int_0^T x\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma}\frac{1}{\sigma S_t^1}\,\mathrm{d}S_t^1 \\ &= x + \int_0^T x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t \frac{1}{1-\gamma}\frac{\mu-r}{\sigma}\frac{1}{\sigma S_t^1}\,\mathrm{d}S_t^1 \\ &= x + \int_0^T \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t \mathrm{d}S_t^1. \end{aligned}$$

This gives the first claim. Now using again that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$ and that $\lambda = \frac{\mu - r}{\sigma}$, we have

$$\begin{aligned} u(x) = E\left[U(f_x^*)\right] &= \frac{x^\gamma}{\gamma} E\left[\left(\mathcal{E}\Big(\frac{\lambda}{1-\gamma}R\Big)_T\right)^\gamma\right] \\ &= \frac{x^\gamma}{\gamma} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right)\right] \\ &= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma}W\right)_T\right] \\ &= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right) \\ &= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right). \end{aligned}$$

This completes the proof.

(d) By part (b) and since $\lambda = \frac{\mu - r}{\sigma}$, we have

$$V_t(x, \vartheta^x) = x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t,$$

and by part (c), we have

$$\vartheta_t^x = \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t.$$

Therefore, we obtain directly that

$$\pi_t^* := \frac{\vartheta_t^x S_t^1}{V_t(x, \vartheta^x)} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.$$

This completes the proof.

Exercise 12.5 *($\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ has moments of all orders)* Let S be a continuous real-valued semimartingale satisfying the structure condition (SC), i.e. there exist a continuous local martingale M null at zero and a predictable process λ such that

$$S = S_0 + M + \int \lambda \, \mathrm{d}\langle M \rangle,$$

and with the mean-variance tradeoff process $K = \int \lambda^2 \, \mathrm{d}\langle M \rangle$ bounded. Now define $\hat{Z} := \mathcal{E}(-\lambda \bullet M)$ and $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \hat{Z}_T$.

(a) Show that $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}(S)$.

(b) Show that both $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $\frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$ have moments of all orders.

Solution 12.5

(a) We need to show that $\hat{P}$ is an equivalent probability measure, and that S is a $\hat{P}$-local martingale. To this end, first note that since K is bounded, we have that

$$E\left[\exp\left(\frac{1}{2}\langle -\lambda \bullet M\rangle_T\right)\right] = E\left[\exp\left(\frac{1}{2}K_T\right)\right] < \infty.$$

So by Novikov's condition, we can conclude that $\hat{Z}$ is a martingale. As $\hat{Z}$ is strictly positive, it follows that $\hat{P}$ is an equivalent probability measure. It now remains to show that S is a $\hat{P}$-local martingale. To this end, we first apply the stochastic product rule to $\hat{Z}S$ and write

$$\mathrm{d}(\hat{Z}S) = \hat{Z}\,\mathrm{d}S + S\,\mathrm{d}\hat{Z} + \mathrm{d}\langle \hat{Z}, S\rangle.$$

Then we use that S satisfies (SC) and that

$$\mathrm{d}\hat{Z} = \mathrm{d}\mathcal{E}(-\lambda \bullet M) = \mathcal{E}(-\lambda \bullet M)\,\mathrm{d}(-\lambda \bullet M) = -\lambda\mathcal{E}(-\lambda \bullet M)\,\mathrm{d}M = -\lambda\hat{Z}\,\mathrm{d}M$$

to compute

$$\begin{aligned}\mathrm{d}(\hat{Z}S) &= \hat{Z}\,\mathrm{d}M + \hat{Z}\lambda\,\mathrm{d}\langle M\rangle - S\lambda\hat{Z}\,\mathrm{d}M - \lambda\hat{Z}\,\mathrm{d}\,\langle M\rangle \\ &= (\hat{Z} - S\lambda\hat{Z})\,\mathrm{d}M.\end{aligned}$$

As $\hat{Z}$, S and M are continuous, it follows that $\hat{Z}S$ is a P-local martingale, so that S is a $\hat{P}$-local martingale, and hence $\hat{P} \in \mathbb{P}_{\mathrm{e,loc}}$, as required.

(b) We compute, for any $p \in \mathbb{R}$,

$$\begin{aligned}\hat{Z}_T^p &= \exp\left(-p\lambda \bullet M_T - \frac{1}{2}p\lambda^2 \bullet \langle M\rangle_T\right) \\ &= \exp\left(-p\lambda \bullet M_T - \frac{1}{2}p^2\lambda^2 \bullet \langle M\rangle_T\right)\exp\left(\frac{1}{2}(p^2-p)\lambda^2 \bullet \langle M\rangle_T\right) \\ &= \mathcal{E}(-p\lambda \bullet M)_T \exp\Big((p^2-p)K_T\Big).\end{aligned}$$

So letting $C < \infty$ be a bound on K, we can write

$$E[\hat{Z}_T^p] \leqslant E[\mathcal{E}(-p\lambda \bullet M)_T]\exp(C|p^2-p|) \leqslant \exp(C|p^2-p|) < \infty,$$

since $\mathcal{E}(-p\lambda \bullet M)$ is a supermartingale. As $Z_T = \frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $Z_T^{-1} = \frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$, this completes the proof.