Mathematical Finance

Exercise Sheet 2

Submit by 12:00 on Wednesday, October 9 via the course homepage.

Exercise 2.1 *(From local martingale to supermartingale)* Let $(X_t)_{t\geqslant 0}$ be a local martingale null at zero and $(Y_t)_{t\geqslant 0}$ a martingale such that $Y_t \leqslant X_t$ for each $t \geqslant 0$. Prove that X is a supermartingale.

Note. This result shows in particular that a local martingale null at zero is a supermartingale if it is bounded below by a constant.

Solution 2.1 Define the difference process $Z := X - Y$. Note that since $X = Z + Y$ and Y is a martingale (and hence a supermartingale), it suffices to show that Z is a supermartingale.

Since a martingale is a local martingale, and the space of local martingales is a vector space, we know that Z is a local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for Z. Since Z is nonnegative, we can apply the Fatou's lemma to get

$$E[Z_t \mid \mathcal{F}_s] \leqslant \liminf_{n\to\infty} E[Z_{t\wedge\tau_n} \mid \mathcal{F}_s] = \liminf_{n\to\infty} Z_{s\wedge\tau_n} = Z_s.$$

Taking expectations of the above with $s = 0$ gives

$$0 \leqslant E[Z_t] \leqslant E[Z_0] = -E[Y_0] < \infty.$$

It follows that Z is integrable and a supermartingale. This completes the proof.

Exercise 2.2 *(Equivalent characterisation of arbitrage)* For a finite time horizon $T > 0$, fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$ and a semimartingale $S = (S_t)_{0\leqslant t\leqslant T}$. Recall the following notations:

- Θ_{adm} is the family of admissible integrands for S.
- $G_T(\Theta_{\mathrm{adm}}) := \{G_T(\vartheta) : \vartheta \in \Theta_{\mathrm{adm}}\}$.
- L^0_+ is the family of (equivalence classes, for P-a.s. equality, of) nonnegative random variables.
- (NA) denotes the "general" absence of arbitrage condition

$$G_T(\Theta_{\mathrm{adm}}) \cap L^0_+ = \{0\}.$$

This says that for any self-financing strategy with zero initial capital, the only way to ensure a nonnegative final value (with probability 1) at expiry is to have value zero at expiry.

- $\mathcal{C}^0_{\text{adm}} := G_T(\Theta_{\text{adm}}) - L^0_+ = \{G_T(\vartheta) - Y : \vartheta \in \Theta_{\text{adm}}, Y \in L^0_+\}$.

Prove that (NA) is equivalent to $\mathcal{C}^0_{\text{adm}} \cap L^\infty \cap L^0_+ = \{0\}$.

Solution 2.2 First assume that (NA) holds. Take $G_T(\vartheta) - Y \in \mathcal{C}^0_{\text{adm}} \cap L^0_+$. Since $G_T(\vartheta) - Y \in L^0_+$ and $Y \in L^0_+$, then also $G_T(\vartheta) \in L^0_+$, and thus by (NA), $G_T(\vartheta) = 0$. We thus have $G_T(\vartheta) - Y = -Y \in L^0_+$, and hence $Y = 0$, so that $G_T(\vartheta) - Y = 0$. It follows that $\mathcal{C}^0_{\text{adm}} \cap L^\infty \cap L^0_+ \subseteq \{0\}$, and since clearly $0 \in \mathcal{C}^0_{\text{adm}} \cap L^\infty \cap L^0_+$, we have equality.

Conversely, take $G_T(\vartheta) \in G_T(\Theta_{\text{adm}}) \cap L^0_+$. We need to show that $G_T(\vartheta) = 0$. Note that for each $n \in \mathbb{N}$,

$$G_T(\vartheta) \wedge n = G_T(\vartheta) - \Big((G_T(\vartheta) - n) \vee 0\Big) \in \mathcal{C}^0_{\text{adm}}.$$

Moreover, $0 \leqslant G_T(\vartheta) \wedge n \leqslant n$ so that $G_T(\vartheta) \in \mathcal{C}^0_{\text{adm}} \cap L^\infty \cap L^0_+$, and hence by assumption $G_T(\vartheta) \wedge n = 0$. Finally, we have

$$G_T(\vartheta) = \lim_{n\to\infty} G_T(\vartheta) \wedge n = 0,$$

as required.

Exercise 2.3 *(Equivalent martingale measure)* Let S be a semimartingale with respect to the probability measure P, and suppose $Q \approx P$ is an equivalent probability measure satisfying $E_Q[Y] \leqslant 0$ for all $Y \in \mathcal{C}^0_{\text{adm}} \cap L^\infty$.

(a) Prove that Q satisfies $E_Q[G_T(\vartheta)] \leqslant 0$ for all $\vartheta \in \Theta_{\text{adm}}$.

(b) If S is bounded, prove that S is a Q-martingale.

Solution 2.3

(a) Let $\vartheta \in \Theta_{\text{adm}}$. For $n \in \mathbb{N}$, note that

$$G_T(\vartheta) \wedge n = G_T(\vartheta) - \Big((G_T(\vartheta) - n) \vee 0\Big) \in \mathcal{C}^0_{\text{adm}}.$$

Since $\theta \in \Theta_{\text{adm}}$, there is some $a > 0$ such that $G_T(\vartheta) \geqslant -a$. Also, we have $G_T(\vartheta) \wedge n \leqslant n$ by construction, and thus $-a \leqslant G_T(\vartheta) \wedge n \leqslant n$ so that $G_T(\vartheta) \wedge n \in L^\infty$. Hence, $G_T(\vartheta) \wedge n \in \mathcal{C}^0_{\text{adm}} \cap L^\infty$. It follows from the assumption that $E_Q[G_T(\vartheta) \wedge n] \leqslant 0$. Fatou's lemma thus gives

$$E_Q[G_T(\vartheta)] \leqslant \liminf_{n\to\infty} E_Q[G_T(\vartheta) \wedge n] \leqslant 0,$$

and hence $E_Q[G_T(\vartheta)] \leqslant 0$, as required.

(b) We are given that S is a semimartingale (with respect to P) and thus adapted. Also, S is integrable since it is bounded. It remains to verify that S satisfies the martingale property. So fix times $0 \leqslant s < t$ and $A \in \mathcal{F}_s$. It suffices to show that

$$E_Q[(S_t - S_s)\mathbf{1}_A] = 0, \tag{1}$$

since this implies that $E_Q[S_t - S_s \mid \mathcal{F}_s] = 0$ by the definition of the conditional expectation.

Now, define the process $\vartheta = \mathbf{1}_{A\times(s,t]}$. Then ϑ is predictable, S-integrable and admissible, since S is bounded. We compute

$$G_T(\vartheta) = \int_0^T \vartheta \, \mathrm{d}S = \mathbf{1}_A(S_t - S_s).$$

With (1) in mind, it remains to show $E_Q[G_T(\vartheta)] = 0$. By part (a), we have $E_Q[G_T(\vartheta)] \leqslant 0$. By considering $-\vartheta \in \Theta_{\mathrm{adm}}$, we also have $E_Q[G_T(-\vartheta)] \leqslant 0$, and since $G_T(-\vartheta) = -G_T(\vartheta)$, this yields $E_Q[G_T(\vartheta)] = 0$, completing the proof.

Exercise 2.4 *(Example of arbitrage on a finite time interval)* Let us consider the arbitrage strategy (with zero initial value) on the infinite time interval $[0, \infty)$ given by

$$\vartheta = \mathbf{1}_{]\!]0,\tau]\!]}, \quad \tau := \inf\{t \geqslant 0 : W_t = 1\},$$

where W is a Brownian motion. Note that for the above strategy, we must be on the infinite time interval $[0, \infty)$ because although $\tau < \infty$ a.s., τ is unbounded.

Construct a similar arbitrage strategy on the interval $[0, T]$, where $T > 0$ is a fixed finite horizon.

Hint: Consider the geometric Brownian motion $\overline{S} = (\overline{S}_t)_{t\geqslant 0}$ given by

$$\overline{S}_t = \exp\left(W_t - \frac{1}{2}t\right),$$

which is adapted to the filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geqslant 0}$. Apply the time-change $t = \tan u$ and let $T = \pi/2$ be the expiry time (after the time change).

Solution 2.4 By the law of the iterated logarithm, we have $\lim_{t\to\infty} \overline{S}_t = 0$. So the stopping time $\bar{\tau} := \inf\{t \geqslant 0 : \overline{S}_t = 1/2\}$ is a.s. finite. Define the process $S = (S_u)_{0\leqslant u\leqslant \pi/2}$ by

$$S_u = \overline{S}_{\tan u}, \qquad 0 \leqslant u < \pi/2,$$

and $S_{\pi/2} = 0$. Then S is adapted to the filtration $\mathbb{F} = (\mathcal{F}_u)_{0\leqslant u\leqslant \pi/2}$ defined by $\mathcal{F}_u = \overline{\mathcal{F}}_{\tan u}$. Note also that since $\lim_{t\to\infty} \overline{S}_t = 0$, the process S is continuous. Define $\tau := \arctan \overline{\tau}$, which is a stopping time with respect to $\mathbb{F}$ (since tan is strictly increasing on $[0, \pi/2]$) satisfying $0 \leqslant \tau < \pi/2$. We define the self-financing strategy

$\varphi \mathrel{\widehat{=}} (0, \vartheta)$ by $\vartheta := -\mathbf{1}_{\rrbracket 0,\tau \rrbracket}$ (note ϑ is adapted and left-continuous, thus predictable). We compute

$$V_t(\varphi) = \int_0^t \vartheta_u \, \mathrm{d}S_u = -\int_0^{t\wedge\tau} \mathrm{d}S_u = S_0 - S_t^\tau = 1 - S_t^\tau.$$

So since $\tau < \pi/2$, we have $V_{\pi/2}(\varphi) = 1 - S_\tau = 1 - \overline{S}_\tau = 1/2$, and thus $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ is an arbitrage strategy.