Mathematical Finance

Exercise Sheet 3

Submit by 12:00 on Wednesday, October 16 via the course homepage.

Exercise 3.1 *(Integrability property of local martingales)* Fix a finite time horizon $T > 0$ and define the space $\mathcal{H}^1$ of martingales by

$$\mathcal{H}^1 := \left\{ M = (M_t)_{0 \leqslant t \leqslant T} : M \text{ RCLL martingale, } M_T^* := \sup_{0 \leqslant t \leqslant T} |M_t| \in L^1 \right\}.$$

Show that every (RCLL) local martingale is locally in $\mathcal{H}^1$. That is, for each local martingale M, show that there is a sequence of stopping times $\tau_n \uparrow T$ stationarily such that $M^{\tau_n} \in \mathcal{H}^1$ for all $n \in \mathbb{N}$.

Solution 3.1 Let M be a local martingale. Then there is a sequence (σ_n) of stopping times with $\sigma_n \uparrow T$ stationarily and such that M^{σ_n} is a martingale for each n. For each n, define the stopping time

$$\rho_n := \inf\{t \geqslant 0 : |M_t| > n\} \wedge T$$

and set $\tau_n := \sigma_n \wedge \rho_n$. Since of course $\rho_n \uparrow T$ stationarily, also $\tau_n \uparrow T$. It thus suffices to show that for fixed $n \in \mathbb{N}$, $M^{\tau_n} \in \mathcal{H}^1$.

We know that M^{σ_n} is a martingale, and since a stopped martingale is a martingale, also $M^{\tau_n} = M^{\sigma_n \wedge \rho_n} = (M^{\sigma_n})^{\rho_n}$ is a martingale. Moreover,

$$(M^{\tau_n})_T^* = \sup_{0 \leqslant t \leqslant \tau_n} |M_t| = \max\left\{ \sup_{0 \leqslant t < \tau_n} |M_t|, |M_{\tau_n}| \right\}. \tag{1}$$

By the construction of ρ_n, we have

$$\sup_{0 \leqslant t < \tau_n} |M_t| = \sup_{0 \leqslant t < \sigma_n \wedge \rho_n} |M_t| \leqslant \sup_{0 \leqslant t < \rho_n} |M_t| \leqslant n.$$

Also, since M^{τ_n} is a martingale, in particular $M_T^{\tau_n} = M_{\tau_n}$ is integrable. It follows immediately from (1) that $(M^{\tau_n})_T^*$ is integrable. This completes the proof.

Exercise 3.2 *(Doob decomposition)* Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$ be a filtered probability space in discrete time, and let $X = (X_k)_{k \in \mathbb{N}_0}$ be a supermartingale.

(a) Prove that there exist a martingale $M = (M_k)_{k\in\mathbb{N}_0}$ and an increasing, integrable and predictable process $A = (A_k)_{k\in\mathbb{N}_0}$ such that

$$X = X_0 + M - A.$$

(b) Prove that if we further impose that M and A are both null at zero, then they are unique up to P-a.s. equality.

Solution 3.2 To simplify notation, we omit (as always) "P-a.s" from all equalities below.

(a) For each $k \in \mathbb{N}_0$, take

$$M_k = \sum_{j=1}^{k}(X_j - E[X_j \mid \mathcal{F}_{j-1}]).$$

It is immediate that M is adapted and integrable. For each $k \in \mathbb{N}$, we have

$$\begin{aligned} E[M_k - M_{k-1} \mid \mathcal{F}_{k-1}] &= E\Big[X_k - E[X_k \mid \mathcal{F}_{k-1}] \Bigm| \mathcal{F}_{k-1}\Big] \\ &= E[X_k \mid \mathcal{F}_{k-1}] - E[X_k \mid \mathcal{F}_{k-1}] \\ &= 0, \end{aligned}$$

and thus M is a martingale. Next, for each $k \in \mathbb{N}_0$, we set

$$\begin{aligned} -A_k = X_k - X_0 - M_k &= X_k - X_0 - \sum_{j=1}^{k}(X_j - E[X_j \mid \mathcal{F}_{j-1}]) \\ &= \sum_{j=1}^{k}(E[X_j \mid \mathcal{F}_{j-1}] - X_{j-1}). \end{aligned}$$

Then A is predictable, and of course $X = X_0 + M - A$, as required. Moreover, A is integrable like X, and A is increasing because

$$A_k - A_{k-1} = X_{k-1} - E[X_k \mid \mathcal{F}_{k-1}] \geqslant 0,$$

by the supermartingale property of X.

(b) Notice that the processes M and A we defined in part (a) are both null at zero, and thus we have existence. To prove uniqueness, suppose the processes $M^{(1)}, A^{(1)}$ and $M^{(2)}, A^{(2)}$ both satisfy the conditions of the problem. Subtracting the equalities

$$\begin{aligned} X - X_0 &= M^{(1)} - A^{(1)}, \\ X - X_0 &= M^{(2)} - A^{(2)} \end{aligned}$$

gives

$$M^{(1)} - M^{(2)} = A^{(1)} - A^{(2)} =: Y.$$

Then Y is predictable like $A^{(1)}$ and $A^{(2)}$, and hence for all $k \in \mathbb{N}$,

$$Y_k = E[Y_k \mid \mathcal{F}_{k-1}].$$

But also Y is a martingale like $M^{(1)}$ and $M^{(2)}$, and hence we get

$$Y_k = Y_{k-1}, \qquad \forall k \in \mathbb{N}.$$

Since $Y_0 = 0$, this implies that $Y_k = 0$ for all $k \in \mathbb{N}_0$, and hence

$$M^{(1)} = M^{(2)} \qquad \text{and} \qquad A^{(1)} = A^{(2)}.$$

This completes the proof.

Exercise 3.3 *(Conditional expectation of increments)* Let $X = (X_t)_{t\geqslant 0}$ be an adapted bounded RCLL process. Prove that for each fixed $t \geqslant 0$,

$$\lim_{u\downarrow t} E[X_u - X_t \mid \mathcal{F}_t] = 0.$$

Can we relax boundedness to a weaker condition?

What can we say for $\lim_{s\uparrow t} E[X_t - X_s \mid \mathcal{F}_s]$, where again $t > 0$ is fixed?

Solution 3.3 By right-continuity of X, we have $\lim_{u\downarrow t}(X_u - X_t) = 0$ P-a.s. By boundedness of X, we can find some $M \in \mathbb{N}$ such that $|X_u| \leqslant M$ for all u. So by the dominated convergence theorem,

$$\lim_{u\downarrow t} E[X_u - X_t \mid \mathcal{F}_t] = E\left[\lim_{u\downarrow t} X_u - X_t \,\middle|\, \mathcal{F}_t\right] = 0, \qquad P\text{-a.s.}$$

We can replace the boundedness condition (which is really a "uniform" boundedness condition over all $\omega \in \Omega$ and $t \geqslant 0$) with the integrability condition

$$E\left[\sup_{t\geqslant 0} |X_t|\right] < \infty.$$

In this case, we can write $|X_u - X_t| \leqslant 2\sup_{t\geqslant 0}|X_t| \in L^1$ and apply the dominated convergence theorem to get the same result. A bit more generally, it is enough if $\sup_{0\leqslant t\leqslant T}|X_t| \in L^1$ for every $T \geqslant 0$. Note also that the above argument does not need X to be adapted.

Now consider the limit $\lim_{s\uparrow t} E[X_t - X_s \mid \mathcal{F}_s]$. For each $s < t$, we have

$$E[X_t - X_s \mid \mathcal{F}_s] = E[X_t \mid \mathcal{F}_s] - X_s$$

because X is adapted. Let $\mathcal{F}_{t-} := \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$ be the smallest σ-field containing $\mathcal{F}_s$ for all $s < t$. Using the fact that by the martingale convergence theorem, for any integrable random variable, its conditional expectation under increasing σ-fields converges to its conditional expectation under the limit σ-field, we see that

$$\lim_{s\uparrow t} E[X_t \mid \mathcal{F}_s] = E[X_t \mid \mathcal{F}_{t-}].$$

Setting $X_{t-} := \lim_{s\uparrow t} X_s$, which is an $\mathcal{F}_{t-}$-measurable random variable, we get

$$\lim_{s\uparrow t} E[X_t - X_s \mid \mathcal{F}_s] = E[X_t - X_{t-} \mid \mathcal{F}_{t-}] = E[\Delta X_t \mid \mathcal{F}_{t-}].$$

This completes the problem.