Coordinator
Chiara Rossato

Mathematical Finance

Exercise Sheet 7

Submit by 12:00 on Wednesday, November 13 via the course homepage.

Exercise 7.1 *(Admissibility at expiry)* Let S be a semimartingale satisfying (NA), and suppose $\vartheta \in \Theta_{\mathrm{adm}}$ has $G_T(\vartheta) \geqslant -a$ P-a.s. for some $a \geqslant 0$. Show that $G(\vartheta) \geqslant -a$ P-a.s., i.e. that ϑ is a-admissible.

Solution 7.1 Since $G(\vartheta)$ is right-continuous, it suffices to show that $G_t(\vartheta) \geqslant -a$ P-a.s. for each $t \in (0, T)$. Suppose for a contradiction that there exists $t \in (0, T)$ with $P[G_t(\vartheta) < -a] > 0$. Consider the integrand ϑ' that waits until after time t to follow ϑ on the event $\{G_t(\vartheta) < -a\}$. That is, we define $\vartheta' := \vartheta \mathbf{1}_{\{G_t(\vartheta)<-a\}\times(t,T]}$. Note that ϑ' is predictable, S-integrable and satisfies

$$G(\vartheta') = \Big(G(\vartheta) - G_t(\vartheta)\Big)\mathbf{1}_{\{G_t(\vartheta)<-a\}\times(t,T]}. \tag{1}$$

In particular, we have

$$G_T(\vartheta') = \Big(G_T(\vartheta) - G_t(\vartheta)\Big)\mathbf{1}_{\{G_t(\vartheta)<-a\}} \geqslant \Big(-a - G_t(\vartheta)\Big)\mathbf{1}_{\{G_t(\vartheta)<-a\}} \in L^0_+ \backslash \{0\}.$$

Since $\vartheta \in \Theta_{\mathrm{adm}}$, there exists some $c \geqslant 0$ such that $G(\vartheta) \geqslant -c$ P-a.s., and hence from (1) we get

$$G(\vartheta') \geqslant -c + a,$$

so that $\vartheta' \in \Theta_{\mathrm{adm}}$. Note that we may assume $c > a$ so that $-c + a = -(c - a)$ has $c - a \geqslant 0$; indeed, if $c \leqslant a$ and $G(\vartheta) \geqslant -c$, then $G(\vartheta) \geqslant -a$ so that ϑ is already a-admissible. We have thus shown that $G_T(\vartheta') \in \mathcal{G}_{\mathrm{adm}} \cap L^0_+ \backslash \{0\}$, which contradicts (NA). This completes the proof.

Exercise 7.2 *(All gains are zero)*

(a) Construct an example where S is a martingale, but $\mathcal{G}_{\mathrm{adm}} = \{0\}$. You may use part (b).

(b) Show that if any continuous adapted process is deterministic, then so is any predictable process.

Solution 7.2

(a) Let $Z \sim \mathcal{N}(0,1)$ be a standard normal random variable, and define the process $S = (S_t)_{0 \leqslant t \leqslant T}$ by

$$S_t = \begin{cases} 0 & \text{if } 0 \leqslant t < T, \\ Z & \text{if } t = T. \end{cases}$$

Clearly S is integrable. Consider the natural filtration $\mathbb{F}^S$. Since S is deterministic for $t \in [0,T)$, then $\mathcal{F}_t^S$ is trivial for $t \in [0,T)$. In particular, for all $t \in [0,T)$ we have

$$E[S_T \mid \mathcal{F}_t^S] = E[S_T] = E[Z] = 0 = S_t.$$

It follows that S is a martingale. It remains to show that $\mathcal{G}_{\mathrm{adm}} = \{0\}$.

Note that since $\mathcal{F}_t^S$ is trivial for each $t \in [0,T)$, any adapted process must be deterministic on $[0,T)$, and thus any adapted and left-continuous process must be deterministic on $[0,T]$. The same then holds for any predictable process (by part (b)).

Now take $\vartheta \in \Theta_{\mathrm{adm}}$. Since ϑ is predictable, it must be deterministic. So let $c := \vartheta_T$. Since S is constant on $[0,T)$, we have

$$G_t(\vartheta) = \begin{cases} 0 & \text{if } 0 \leqslant t < T, \\ cS_T & \text{if } t = T. \end{cases} \tag{2}$$

In particular, $G_T(\vartheta) = cZ \sim \mathcal{N}(0,c^2)$ is unbounded unless $c = 0$ (in which case $G_T(\vartheta) \equiv 0$). It follows from (2) that $G_T(\vartheta) \in \mathcal{G}_{\mathrm{adm}}$ if and only if $c = 0$, which implies $\mathcal{G}_{\mathrm{adm}} = \{0\}$, as required.

(b) Recall the monotone class theorem for functionals:

Fix a set E, and let $B(E)$ denote the family of bounded functionals $f : E \to \mathbb{R}$. Suppose $\mathcal{H} \subseteq B(E)$ is a linear subspace of $B(E)$ containing the constant function 1 and satisfying the following condition:

if $f_1, f_2, \ldots \in \mathcal{H}$ with $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots$ and $f := \lim_{n\to\infty} f_n \in B(E)$, then $f \in \mathcal{H}$.

Then for any subset $\mathcal{K} \subseteq \mathcal{H}$ that is closed under multiplication (i.e. if $f, g \in \mathcal{K}$ then $fg \in \mathcal{K}$), $\mathcal{H}$ contains all bounded $\sigma(\mathcal{K})$-measurable functionals.

In the theorem above, we take $E := \Omega \times [0,T]$, so that $B(E)$ denotes the family of bounded processes. Let $\mathcal{H} \subseteq B(E)$ be the subspace of bounded deterministic processes. Clearly $\mathcal{H}$ satisfies the conditions of the theorem. Next take $\mathcal{K}$ to be the family of all continuous and adapted processes. By assumption, these processes are deterministic and hence also bounded, so that $\mathcal{K} \subseteq \mathcal{H}$. Since $\mathcal{K}$ is closed under multiplication, the monotone class theorem implies that $\mathcal{H}$ contains all bounded $\sigma(\mathcal{K})$-measurable functionals. That is, all bounded predictable processes are deterministic. To conclude that any predictable process is deterministic, simply take some predictable X, and note that $X := \lim_{n\to\infty} X \wedge n$ is the (pointwise) limit of bounded predictable processes. This completes the proof.

Exercise 7.3 *(From σ-martingale to local martingale)* Argue in detail that every continuous σ-martingale null at zero is a local martingale null at zero.

Can you find an example where it is not a supermartingale?

Solution 7.3 Let X be a continuous σ-martingale null at 0, so that $X = \psi \bullet M$ for a d-dimensional local martingale $M = (M^i)_{1\leqslant i\leqslant d}$ and a positive one-dimensional predictable integrand $\psi \in L(M)$. Define the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n := \inf\{t \geqslant 0 : |X_t|_\infty \geqslant n\},$$

where $|\cdot|_\infty$ denotes the supremum norm on $\mathbb{R}^d$ (i.e. $|(x_1,\dots,x_d)|_\infty = \max_i |x_i|\}$). Since X is RCLL, it is bounded on compact intervals (with the bounded depending on the trajectory $X_\cdot(\omega)$, hence on ω), and thus $\tau_n \uparrow T$ stationarily. Moreover, since X is null at zero, we have for each $n \in \mathbb{N}$ that

$$X^{\tau_n} = \mathbf{1}_{]\!]0,\tau_n]\!]} \bullet X = \mathbf{1}_{]\!]0,\tau_n]\!]} \bullet (\psi \bullet M) = (\mathbf{1}_{]\!]0,\tau_n]\!]}\psi) \bullet M.$$

In particular, X^{τ_n} is a stochastic integral against the local martingale M. Moreover, X^{τ_n} is bounded (by n) because X is continuous, and so the Ansel–Stricker theorem implies that X^{τ_n} is a local martingale. As bounded local martingales are martingales, we have that X^{τ_n} is martingale for each $n \in \mathbb{N}$, and hence X is a continuous local martingale null at zero. (Alternatively, one can use that $(\mathcal{M}_{0,\mathrm{loc}})_{\mathrm{loc}} = \mathcal{M}_{0,\mathrm{loc}}$.)

To find an example where X is not a supermartingale, take M to be any local martingale that is not a supermartingale (e.g. $-S$ from Exercise 4.1), and then let $\psi \equiv 1 \in L(M)$. Then $X = M - M_0$, which is not a supermartingale by assumption.

Exercise 7.4 *(Theorem 4.5)* Let S be a semimartingale. Prove (3) $\Longrightarrow$ (1) in Theorem 4.5, i.e. the existence of an equivalent σ-martingale measure for S implies (NFLVR).

Solution 7.4 We give two proofs. The first shows that S satisfies (NFLVR) under Q directly using the definition. The second takes advantage of Proposition 4.3 and shows that S satisfies (NUPBR) instead.

Solution 1. We need to show $\overline{\mathcal{C}}^{L^\infty} \cap L^\infty_+ = \{0\}$, where we recall $\mathcal{C} := (\mathcal{G}_{\mathrm{adm}} - L^0_+) \cap L^\infty$. So take some $f \in \overline{\mathcal{C}}^{L^\infty} \cap L^\infty_+$. Then there exists a sequence $(f_n) \subseteq \mathcal{C}$ such that $f_n \to f$

in L^∞. As $f_n \in \mathcal{C}$, there exists some $g_n \in \mathcal{G}_{\text{adm}}$ such that $g_n - f_n \in L^0_+$. Since $f \in L^\infty_+$, then for each $n \in \mathbb{N}$, we have

$$-\|f_n - f\|_{L^\infty} \leqslant f_n \leqslant g_n. \tag{3}$$

Now suppose Q is an equivalent σ-martingale measure for S. In particular, Q is an equivalent separating measure for S, and thus $E_Q[g_n] \leqslant 0$ for each $n \in \mathbb{N}$. So the Fatou lemma together with (3) (which implies that the f_n are uniformly bounded from below) gives

$$E_Q[f] \leqslant \liminf_{n\to\infty} E_Q[f_n] \leqslant 0.$$

As $f \in L^\infty_+$, we conclude that $f = 0$ Q-a.s. and hence also $f = 0$ P-a.s., as required.

Solution 2. By Proposition 4.3, it suffices to show that S satisfies (NA) and (NUPBR) under Q. To show that S satisfies (NA), take $\vartheta \in \Theta_{\text{adm}}$ with $G_T(\vartheta) \in L^0_+$. Then $G_T(\vartheta) \geqslant 0$ Q-a.s, and since Q is an equivalent separating measure for S, we have $E_Q[G_T(\vartheta)] \leqslant 0$. This implies $G_T(\vartheta) = 0$ Q-a.s., and hence $G_T(\vartheta) = 0$ P-a.s. Hence S satisfies (NA).

It remains to show that S satisfies (NUPBR) under Q, i.e. that

$$\lim_{n\to\infty} \sup_{\vartheta\in\Theta^1} Q[|G_T(\vartheta)| \geqslant n] = 0.$$

To this end, note that for each $\vartheta \in \Theta^1$ and integer $n \geqslant 2$, the 1-admissibility of ϑ gives

$$Q[|G_T(\vartheta)| \geqslant n] = Q[G_T(\vartheta) \geqslant n],$$

and since $G_T(\vartheta) + 1 \geqslant 0$ Q-a.s., we can apply Markov's inequality to get

$$Q[|G_T(\vartheta)| \geqslant n] \leqslant \frac{1}{n+1} E_Q[G_T(\vartheta) + 1].$$

Again using that Q is an equivalent separating measure for S, we have $E_Q[G_T(\vartheta)] \leqslant 0$, and hence

$$Q[|G_T(\vartheta)| \geqslant n] \leqslant \frac{1}{n+1}.$$

We thus have

$$\sup_{\vartheta\in\Theta'} Q[|G_T(\vartheta)| \geqslant n] \leqslant \frac{1}{n+1} \longrightarrow 0 \quad \text{as } n \to \infty,$$

which gives the claim.