Mathematical Finance

Exercise Sheet 8

Submit by 12:00 on Wednesday, November 20 via the course homepage.

Exercise 8.1 *(Uniqueness of the numéraire portfolio)*

(a) Recall Jensen's inequality: *if X is an integrable random variable taking values in an interval $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ is a convex function such that $f(X)$ is also integrable, then*

$$E[f(X)] \geqslant f(E[X]).$$

Show that if f is strictly convex and X is not almost surely constant (i.e., there exists no $c \in \mathbb{R}$ with $P[X = c] = 1$), we have the strict inequality

$$E[f(X)] > f(E[X]).$$

(b) By using part (a) or otherwise, show that there is at most one numéraire portfolio.

Solution 8.1

(a) By definition, f is strictly convex means that for any distinct $x_1, x_2 \in I$, we have the strict inequality

$$f\Big(tx_1 + (1-t)x_2\Big) < tf(x_1) + (1-t)f(x_2), \quad \forall t \in (0,1).$$

On the event $\{X \neq E[X]\}$, we have

$$f\Big(tX + (1-t)E[X]\Big) < tf(X) + (1-t)f(E[X]), \quad \forall t \in (0,1).$$

As the above always holds with weak inequality and $P[X \neq E[X]] > 0$, we can take the expectation of both sides while preserving the strict inequality to get

$$E\Big[f(tX + (1-t)E[X])\Big] < tE[f(X)] + (1-t)f(E[X]), \quad \forall t \in (0,1).$$

Applying Jensen's inequality to the left-hand side gives

$$f(E[X]) < tE[f(X)] + (1-t)f(E[X]), \quad \forall t \in (0,1).$$

Rearranging gives $f(E[X]) < E[f(X)]$, as required.

(b) Recall that an element $X \in \mathcal{X}^1_{++}$ is a numéraire portfolio if for all elements $V \in \mathcal{X}^1_{++}$, the quotient V/X is a supermartingale. So suppose $X, Y \in \mathcal{X}^1_{++}$ are numéraire portfolios. Then both X/Y and Y/X are supermartingales with $X_0 = Y_0 = 1$, and so for all $t \in [0, T]$,

$$E[X_t/Y_t] \leqslant E[X_0/Y_0] = 1 \quad \text{and} \quad E[Y_t/X_t] \leqslant E[Y_0/X_0] = 1.$$

Note that the function $x \mapsto 1/x$ is convex on $(0, \infty)$, and thus Jensen's inequality gives

$$1 \geqslant E[Y_t/X_t] \geqslant 1/E[X_t/Y_t] \geqslant 1,$$

which implies that $E[X_t/Y_t] = 1$ (and $E[Y_t/X_t] = 1$) for all $t \in [0, T]$. In particular, for each $t \in [0, T]$ we have

$$E[Y_t/X_t] = 1/E[X_t/Y_t].$$

As the function $x \mapsto 1/x$ is strictly convex on $(0, \infty)$, this implies that for each $t \in [0, T]$, the random variable X_t/Y_t is almost surely constant. Since $E[X_t/Y_t] = 1$, we obtain $X_t/Y_t = 1$ almost surely for all $t \in [0, T]$, and hence X and Y are modifications of each other. As X and Y are both right-continuous, they are even indistinguishable. This completes the proof.

Exercise 8.2 *(Finding the numéraire portfolio)* Show that if Z is an EσMD for S and $1/Z \in \mathcal{X}^1_{++}$, then $1/Z$ is the numéraire portfolio.

Solution 8.2 Take some $V \in \mathcal{X}^1_{++}$. We need to show that $V/(1/Z) = ZV$ is a supermartingale. To this end, we write $V = 1 + G(\vartheta)$ for some $\vartheta \in \Theta^1$ and apply the stochastic product rule to $ZG(\vartheta)$ to get

$$\begin{aligned} \mathrm{d}\Big(ZG(\vartheta)\Big) &= Z_- \,\mathrm{d}G(\vartheta) + G_-(\vartheta)\,\mathrm{d}Z + \mathrm{d}[Z, G(\vartheta)] \\ &= Z_-\vartheta\,\mathrm{d}S + G_-(\vartheta)\,\mathrm{d}Z + \vartheta\,\mathrm{d}[Z, S]. \end{aligned}$$

Again by the stochastic product rule, we have

$$\mathrm{d}(ZS) = Z_-\,\mathrm{d}S + S_-\,\mathrm{d}Z + \mathrm{d}[Z, S],$$

and therefore $\mathrm{d}[Z, S] = \mathrm{d}(ZS) - Z_-\,\mathrm{d}S - S_-\,\mathrm{d}Z$. We thus have

$$\begin{aligned} \mathrm{d}\Big(ZG(\vartheta)\Big) &= Z_-\vartheta\,\mathrm{d}S + G_-(\vartheta)\,\mathrm{d}Z + \vartheta\,\mathrm{d}(ZS) - \vartheta Z_-\,\mathrm{d}S - \vartheta S_-\,\mathrm{d}Z \\ &= G_-(\vartheta)\,\mathrm{d}Z + \vartheta\,\mathrm{d}(ZS) - \vartheta S_-\,\mathrm{d}Z \\ &= \Big(G_-(\vartheta) - \vartheta S_-\Big)\,\mathrm{d}Z + \vartheta\,\mathrm{d}(ZS). \end{aligned}$$

As Z is an EσMD for S, ZS is a σ-martingale and thus an integral against a local martingale. Since Z is also a local martingale (as it is an EσMD), we have that

$ZG(\vartheta)$ is an integral against a (multi-dimensional) local martingale, and thus so is $ZG(\vartheta) + Z = ZV$. As $Z > 0$ and $V = 1 + G(\vartheta) > 0$, we have $ZV > 0$, and hence the Ansel–Stricker theorem implies that ZV is a supermartingale, as required.

Exercise 8.3 *(Yor's formula)* Recall that for a semimartingale X, the *stochastic exponential* of X, denoted by $\mathcal{E}(X)$, is the unique solution Z to the SDE

$$\mathrm{d}Z = Z_- \,\mathrm{d}X, \quad Z_0 = 1.$$

Prove that for two semimartingales X and Y, the following equality holds

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Solution 8.3 By definition of the stochastic exponential, it suffices to show that $\mathcal{E}(X)\mathcal{E}(Y)$ satisfies the SDE

$$\mathrm{d}Z = Z_- \,\mathrm{d}(X + Y + [X, Y]), \quad Z_0 = 1.$$

As a first step, we clearly have $\mathcal{E}(X)_0\mathcal{E}(Y)_0 = 1$. Next, we apply the stochastic product rule to get

$$\mathrm{d}\Big(\mathcal{E}(X)\mathcal{E}(Y)\Big) = \mathcal{E}(X)_- \,\mathrm{d}\mathcal{E}(Y) + \mathcal{E}(Y)_- \,\mathrm{d}\mathcal{E}(X) + \mathrm{d}[\mathcal{E}(X), \mathcal{E}(Y)].$$

By definition of the stochastic exponential, we have that $\mathrm{d}\mathcal{E}(X) = \mathcal{E}(X)_- \,\mathrm{d}X$ and $\mathrm{d}\mathcal{E}(Y) = \mathcal{E}(Y)_- \,\mathrm{d}Y$. Substituting these values into the above equality gives

$$\begin{aligned}\mathrm{d}\Big(\mathcal{E}(X)\mathcal{E}(Y)\Big) &= \mathcal{E}(X)_-\mathcal{E}(Y)_- \,\mathrm{d}Y + \mathcal{E}(Y)_-\mathcal{E}(X)_- \,\mathrm{d}X + \mathcal{E}(X)_-\mathcal{E}(Y)_- \,\mathrm{d}[X, Y] \\ &= \mathcal{E}(X)_-\mathcal{E}(Y)_- \,\mathrm{d}(X + Y + [X, Y]).\end{aligned}$$

This completes the proof.

Exercise 8.4 *(Digital option)* In the Black–Scholes model, consider the *digital option* with undiscounted payoff $\widetilde{H} = \mathbf{1}_{\{\widetilde{S}_T > \widetilde{K}\}}$, where $\widetilde{K} > 0$ is fixed. Calculate the arbitrage-free price process and the replicating strategy of the digital option, and thus conclude that it is attainable.

Solution 8.4 Let Q denote the EMM for S, and W^Q the corresponding Q-Brownian motion so that $S_t = S_0 \exp(\sigma W_t^Q - \frac{1}{2}\sigma^2 t)$ for all $t \in [0, T]$. Note this also implies that $S_T = S_t \exp(\sigma(W_T^Q - W_t^Q) - \frac{1}{2}\sigma^2(T - t))$. To compute the arbitrage-free price of the digital option at time t, we start by applying the risk-neutral valuation formula to get

$$\begin{aligned}V_t &= E_Q[H \mid \mathcal{F}_t] = e^{-rT} E_Q[\mathbf{1}_{\{S_T > K\}} \mid \mathcal{F}_t] = e^{-rT} Q[S_T > K \mid \mathcal{F}_t] \\ &= e^{-rT} Q\left[S_t \exp\left(\sigma(W_T^Q - W_t^Q) - \frac{1}{2}\sigma^2(T-t) \right) > K \,\middle|\, \mathcal{F}_t \right] \\ &= e^{-rT} Q\left[x \exp\left(\sigma(W_T^Q - W_t^Q) - \frac{1}{2}\sigma^2(T-t) \right) > K \,\middle|\, \mathcal{F}_t \right] \Bigg|_{x=S_t} \\ &= e^{-rT} Q\left[\sigma(W_T^Q - W_t^Q) > \log \frac{K}{x} + \frac{1}{2}\sigma^2(T-t) \right] \Bigg|_{x=S_t}.\end{aligned}$$

As W^Q is a Q-Brownian motion, we can write

$$\sigma(W_T^Q - W_t^Q) = \sigma\sqrt{T-t}\, Z,$$

where $Z \sim \mathcal{N}(0,1)$ under Q. We thus have

$$\begin{aligned}V_t &= e^{-rT} Q\left[Z > \frac{\log \frac{K}{x} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right] \Bigg|_{x=S_t} \\ &= e^{-rT} \Phi\left(\frac{\log \frac{x}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right) \Bigg|_{x=S_t},\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. Let us write

$$v(t,x) := e^{-rT} \Phi\left(\frac{\log \frac{x}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right),$$

so that $v(t, S_t) = V_t$. Applying Itô's formula to $v(t, S_t)$ allows us to write V_t as the sum of an integral against t and a stochastic integral against S_t. As V is a Q-martingale, it must be the case that the integral against t is zero. Moreover, we can see that the integrand of the integral against S_t is

$$\vartheta_t := \frac{\partial v}{\partial x}(t, S_t) = \frac{e^{-rT}}{\sigma\sqrt{T-t}} \phi\left(\frac{\log \frac{S_t}{K} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \right) \frac{1}{S_t},$$

where ϕ denotes the probability density function of the standard normal distribution. As $V \geqslant 0$, then

$$V = V_0 + \int \vartheta \,\mathrm{d}S \geqslant 0,$$

and so the digital option is attainable, with the replicating strategy given by (V_0, ϑ). In undiscounted units, we can write

$$\widetilde{V}_t = e^{rt} V_t = e^{rt} v(t, S_t) = e^{rt} v(t, e^{-rt}\widetilde{S}_t) =: \widetilde{v}(t, \widetilde{S}_t),$$

where $\widetilde{v}(t,x) := e^{rt}v(t, e^{-rt}x)$. The replicating is now $(V_0, \widetilde{\vartheta})$, where

$$\widetilde{\vartheta} = \frac{\partial \widetilde{v}}{\partial x}(t, \widetilde{S}_t) = \frac{\partial v}{\partial x}(t, S_t) = \vartheta.$$

This completes the proof.