Linear Tools for Engineers

Alexander Caspar

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Contents

Introduction

This monograph introduces more advanced Linear Algebra with applications relevant to linear systems of first-order Ordinary Differential Equations (ODE) and Fourier theory. Its primary goal is to equip engineering students with the mathematical tools necessary to model, analyse, and solve practical engineering problems. Understanding mathematics as a precise language allows for accurate modelling of complex systems and provides a framework for interpreting solutions both qualitatively and quantitatively.

By engaging with this material, students will develop a foundational understanding of linear algebraic concepts pivotal in engineering disciplines. This knowledge provides the essential mathematical background required for further study in advanced topics such as partial differential equations, control theory, and signal processing. The Linear Algebra skills gained here will serve as indispensable tools in these areas, facilitating the understanding and application of more advanced mathematical techniques.

Prerequisites Students are expected to have a solid foundation in basic matrix algebra, systems of linear equations, and the fundamental concepts of eigenvalues and eigenvectors. These topics are thoroughly covered in a first year undergraduate course in an engineering Bachelor programme[1]. Additionally, some familiarity with linear ODEs of the first and second order will be beneficial. For those who may need a refresher, an appendix is provided to review these essential concepts.

Students should know appropriate CAS routines and apply them. They should also be able to interpret and analyse CAS code effectively in the context of engineering applications.

Learning outcomes and mathematical topics Upon successful completion of this course, students will be able to apply Linear Algebra concepts in various situations. We give an overview:

Vector spaces and linear maps Students can verify whether a subset is a vector space, understand the concept of basis, coordinates and dimension, and apply these ideas to solve practical problems. Understand and apply linear maps, including the ability to find their matrices in different bases and determine their properties, e.g. diagonalisation.

Linear ODE-systems Students can determine the matrix A in a modelling ODE-system $y' = Ay + g$ and know what a solution of Linear ODE-system, know what a stationary solution is and compute it. Moreover they can compute a solution of Linear ODE-system with eigenvectors and eigenvalues, in the diagonalisable case and with the exponential of a matrix.

Exponential of a matrix Students know what the exponential of a matrix is and know the application to a system. They can apply basic facts about the

[1] For example in David C. Lay et al, Linear Algebra and its Applications, 5/E, Pearson Hall, 2016, Chapters 1 to 3, Chapter 5.1, 5.2, and 6.1, as well as in Appendix B.

exponential and can compute a the exponential for several types of matrices, including of a Jordan block.

Non linear systems Students can compute stationary solution of a non linear system and can apply the concept of linearisation to determine (non) stability of a system.

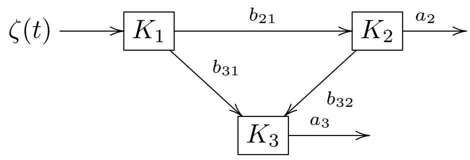
Fourier Theory Students know what a periodic function is and how to compute its continuation. They know the formulae for real and complex Fourier coefficients and how to uses this solving ODEs.

Euclidean Vector Spaces Students know the concepts of norm, unit vectors, and orthogonal vectors and their applications, e.g. compute the coordinates with respect to an orthonormal basis. Compute the projection of a vector onto a subspace and understand the connection with Fourier theory.

Part I

Vector spaces and linear ODE-systems

We start with an example of a typical situation, given a Linear System of Compartments. Consider three compartments (e.g. machines or organs) that are linked. We model this in schematic way by the following :



The function ζ indicates an external impulse or (energy) source that had an impact on the system. In $K_{2/3}$ the coefficient $a_{2/3}$ stands for some kind of (energy) drain.

The function y_i gives the magnitude at the time t in compartment K_i by the value $y_i(t)$. Moreover, we assume that there is a linear interaction between compartment K_i and compartment K_j by $y_i'(t) = \sum_j \pm b_{ij} y_j(t)$. The left hand side $y_i'(t)$ is the value of the derivative of the function y_i at the time t. We want to see the derivative as a measure of alteration in the compartment K_i. If we apply this convention to the above example we get three equations:

$$\left\{ \begin{array}{lcl} y_1'(t) & = & \zeta(t) - b_{21}\ y_1(t) - b_{31}\ y_1(t) \\ y_2'(t) & = & b_{21}\ y_1(t) - b_{32}\ y_2(t) - a_2\ y_2(t) \\ y_2'(t) & = & b_{31}\ y_1(t) + b_{32} y_2(t) - a_2\ y_3(t) \end{array} \right\}.$$

Exercise Check this.

We want to write these equations in a compact form $y'(t) = Ay(t) + g(t)$ by using the notion of the matrix-vector-product from Linear Algebra. On the left hand side we have the vector $y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix}$ and on the right hand side the product of $A = \begin{pmatrix} -(b_{31} + b_{21}) & 0 & 0 \\ b_{21} & -(a_2 + b_{32}) & 0 \\ b_{31} & b_{32} & -a_3 \end{pmatrix}$ with $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$. To complete there is the summand $g(t) = \begin{pmatrix} \zeta(t) \\ 0 \\ 0 \end{pmatrix}$.

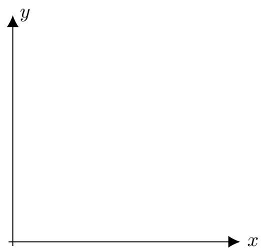
Exercise Check that A and g do the job, i.e. the equation $y'(t) = Ay(t) + g(t)$ is a compact form of the three equations.

Our main task is to understand, which functions y_i fulfil these equations simultaneously? In the above example a solution would be a function $y : \mathbb{R} \to \mathbb{R}^3$ with a graph that is represented by a space curve. Given a matrix A can we write down solution functions? Can we find criteria how such solutions would look like? Or more sophisticated, we can ask how the vector space of solutions looks like. Can we say anything about the structure of the set $\mathcal{L}_A = \{y : \mathbb{R} \to \mathbb{R}^n \mid y' = Ay\}$? To this extend we have to investigate the notion of a vector space.

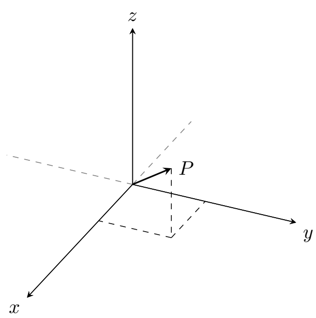
Chapter 1

Vector spaces

Let us start with recalling the well known vector space $\mathbb{R}^2$ and $\mathbb{R}^3$:



The plane $\mathbb{R}^2$



The space $\mathbb{R}^3$

We know how to add two vectors and multiply a vector by a scalar. More general we have also the definition of $\mathbb{R}^n$ for $n > 3$ by the following

1. We write vector v as $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ or $v = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^n$.

2. The sum of $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $w = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ is $v + w = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} \in \mathbb{R}^n$

3. The multiplication by a scalar λ is defined as $\lambda \cdot v = \begin{pmatrix} \lambda \cdot a_1 \\ \lambda \cdot a_2 \\ \vdots \\ \lambda \cdot a_n \end{pmatrix} \in \mathbb{R}^n$.

4. There is a bunch of rules fulfilled by $+$ and $\cdot$.

The idea of an abstract vector space is to generalise those rules that are fulfilled in the vector space $\mathbb{R}^n$.

Definition. *Let V be (a nonempty) set with 2 operations*

$$+ \;:\; V \times V \to V, \quad (v,w) \mapsto v + w \text{ addition of vectors}$$
$$\cdot \;:\; \mathbb{R} \times V \to V, \quad (\lambda, v) \mapsto \lambda v \quad \text{multiplication by scalar}$$

such that for each $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$

- $v + u = u + v$
- $u + (v + w) = (u + v) + w$
- $\lambda(u + v) = \lambda u + \lambda v$ *and* $(\lambda + \mu)v = \lambda v + \mu v$
- $(\lambda\mu)v = \lambda(\mu v)$
- $1 \cdot v = v$
- *There is a (unique) zero vector $0 \in V$ s.t. $u + 0 = u$.*
- *There is a (unique) additive inverse $-u \in V$ s.t. $u + (-u) = 0$.*

We call V a real vector space (VS). If one replaces the real numbers $\mathbb{R}$ with the complex numbers $\mathbb{C}$ one gets a complex VS.

Examples and Discussions For each of the following sets we are looking for the operations $+$ and $\cdot$, that make it to a VS in a natural (obvious) way.

1. $M_{m\times n} = \{$ Matrices with m rows and n columns$\}$:

   Fill in the dots.

   $+$ For $A = (a_{ij}), B = (b_{ij}) \in M_{m\times n} \rightsquigarrow A \pm B = C = (c_{ij}) = \ldots$

   $\cdot$ Scalar λ and $A = (a_{ij}) \in M_{m\times n} \rightsquigarrow \lambda \cdot A = \ldots$

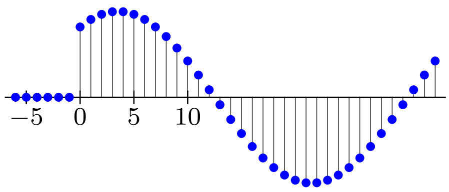
2. $\mathbb{C}^n$ as a complex VS:

   Let $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{C}^n$, where each a_i is a complex number. We add two vectors by adding the coordinates as complex numbers. Also we multiply v with a complex $\lambda \in \mathbb{C}$ in each coordinate. Note that $\overline{v} = \begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \overline{a_n} \end{pmatrix} \in \mathbb{C}^n$ with the complex conjugated in each coordinate.

   We get $\mathbb{C}^n$ as a real VS, in case we restrict the scalar multiplication to real numbers $\lambda \in \mathbb{R} \subset \mathbb{C}$.

3. VS of (discrete-time) signals $\mathbb{S}$:

   The vectors are sequences of numbers $(y_k) = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$. Each coordinate might represent a signal



4. $\mathcal{P}_{\leq n} = \{\text{Polynomials of degree} \leq n\}$
   Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ and
   $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_{n-1}x^{n-1} + b_nx^n$,

   then $(p+q)(x) = c_0 + c_1x + c_2x^2 + \ldots + c_{n-1}x^{n-1} + c_nx^n$ with $c_i = \ldots$

   Question: Why do we assume $\leq n$ and not just $= n$?

5. Set of all functions $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$
   Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with $x \mapsto f(x)$ and $x \mapsto g(x)$. Then

   $$f \underbrace{+}_{\text{new}} g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto (f \underbrace{+}_{\text{new}} g)(x) = \ldots$$

   and with $\lambda \in \mathbb{R}$ we have

   $$\lambda \underbrace{\cdot}_{\text{new}} f : \mathbb{R} \to \mathbb{R} \quad x \mapsto (\lambda \underbrace{\cdot}_{\text{new}} f)(x) = \ldots$$

Examples of function spaces The last example is one of our main applications. In general it collects all functions. As we know from Calculus those function might have additional properties. If we restrict to such a subset the operations on $\mathcal{F}(\mathbb{R}, \mathbb{R})$ are still valid in this subset. Popular examples are

$$C^n(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{The } n\text{-th derivative } f^{(n)} \text{ exists and is continuous}\}$$

and $C^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{All derivatives of } f \text{ exist}\}$. If we consider n functions $f_1 f_2, \ldots f_n$ in $C^1(\mathbb{R})$, they form a vector valued map

$$f : \mathbb{R} \to \mathbb{R}^n, \ t \mapsto f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

All these elements form the VS $C^1(\mathbb{R}, \mathbb{R}^n)$.

Exercise Let $A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{13}{10} \end{pmatrix}$. It defines the equation $y' = Ay$. Verify that the functions $t \mapsto e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $t \mapsto e^{\frac{1}{2}t} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \in C^1(\mathbb{R}, \mathbb{R}^2)$ and also their sum are solutions of $y' = Ay$, i.e. $y'(t) = Ay(t)$ for all t.

Exercises on Signals To familiarise further with the abstract notion of vector space solve the following.

1. Let $y_k = \cos\left(\frac{\pi}{4} \cdot k\right)$.

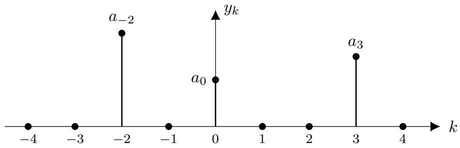
   (a) Find the values $(y_k) = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$ and plot the values in a (k, y_k)-coordinate system.

(b) Compute the signal (z_k) with $z_k = a_k y_k + a_{k+1} y_{k+1} + a_{k+2} y_{k+2}$, where $a_k = a_{k+2} = \frac{\sqrt{2}}{4}$ and $a_{k+1} = \frac{1}{2}$.

(c) Take the signal with $w_k = \cos\left(\frac{3\pi}{4} \cdot k\right)$ and compute the values $a_k w_k + a_{k+1} w_{k+1} + a_{k+2} w_{k+2}$. How do you interpret your result?

2. Define $\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$. **Question:** How would you call this signal?

   (a) Let y_k be defined as in the picture.



   Write y_k in terms of δ_k.

   (b) Try to find a general formula that expresses an arbitrary y_k in terms of δ_k.

1.1 Subspaces

Definition. *Let V be a vector space. A nonempty subset $U \subset V$ is called* **subspace** *of V, if for all $u, v \in U$ and all $\lambda \in \mathbb{R}$*

- $u + v \in U$ *and* $\lambda \cdot u \in U$

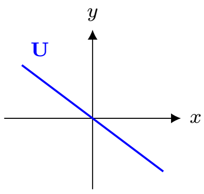
In other words: U itself is a VS that is closed with respect to $+$ and $\cdot$.

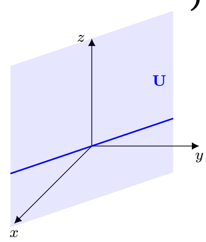
Exercise Why is the zero vector 0 always in a subspace U?

Popular Examples

1. In plane and in space we have for example

$$U = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ with } 3x + 4y = 0 \right\} \quad U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ with } 3x + 4y = 0 \right\}$$





2. We have a sequence of subspaces:

$$\mathcal{F}(\mathbb{R},\mathbb{R}) \supset C^0(\mathbb{R}) \supset C^1(\mathbb{R}) \supset C^2(\mathbb{R}) \supset \dots$$
$$\dots \supset C^\infty(\mathbb{R}) \supset \mathcal{P}_{\leq\infty} \supset \dots \supset \mathcal{P}_{\leq 2} \supset \mathcal{P}_{\leq 1} \supset \mathcal{P}_{\leq 0} \supset \{0\}$$

Exercises

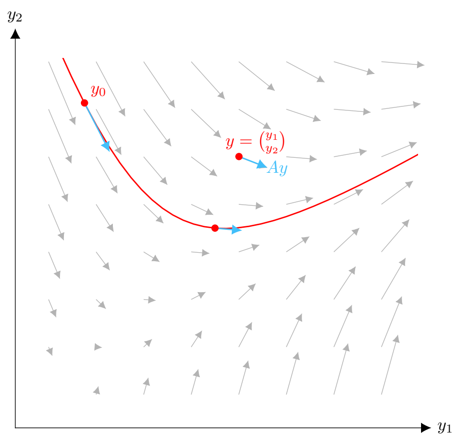
1. Decide whether the given subset $U \subset V$ is a subspace.
   (a) $V = \mathbb{R}^2$ and $U = \left\{ \begin{pmatrix} s \\ t \end{pmatrix} | s \cdot t \geq 0 \right\}$
   (b) $V = \mathbb{R}^3$ with
   $U_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} ; s \in \mathbb{R} \right\}$ and $U_2 = \left\{ \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} ; s \geq 0 \right\}$
   (c) $V = \mathbb{R}^4$ and $U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, t \in \mathbb{R} \right\}$
   (d) $V = \mathcal{P}_{\leq n} = \{\text{Polynomials of degree} \leq n\}$ with
   $U = \dots$
      i. All polynomials of the form $p(x) = ax^2$ where $a \in \mathbb{R}$.
      ii. All polynomials of the form $p(x) = a + x^2$ where $a \in \mathbb{R}$.
      iii. All polynomials in $\mathcal{P}_{\leq 3}$ with integer coefficients.
      iv. All polynomials with $p(0) = 0$.
   (e) $V = M_{2\times 2}$ with $U_1 = \left\{ \begin{pmatrix} a & 0 \\ a^2 & a \end{pmatrix} ; a \in \mathbb{R} \right\}$, $U_2 = \left\{ A | A^\top = -A \right\}$ and $U_3 = \{ \text{ All regular matrices } \}$.
   (f) $V = \mathcal{F}(\mathbb{R},\mathbb{R})$ with
   $U_1 = \{f \in V | f(0) = 1\}$ and $U_2 = \{f \in V | f(1) + f(3) + f(10) = 0\}$
2. Let A be a $m \times n$-matrix and $b \in \mathbb{R}^?$ defining a system $Ax = b$.
   (a) Replace the question mark.
   (b) Let $b = 0$ the zero vector.
   Verify that $\mathcal{K} = \{x | Ax = 0\}$ is a subspace of $\mathbb{R}^?$. Is it true for $b \neq 0$?
   (c) Check that $\mathcal{I}_A = \{b | Ax = b, \text{ for some } x\}$ is a subspace.
   (d) Clarify, where these are subspaces and where x and b lie.

The solution space $\mathcal{L}_A$ A matrix $A \in M_{n\times n}$ defines a system $\boxed{y' = Ay}$. The space of solution is $\mathcal{L}_A = \{y \in C^1(\mathbb{R},\mathbb{R}^n) \mid y' = Ay\}$ and this a subspace of $C^1(\mathbb{R},\mathbb{R}^n)$. **Check that $\mathcal{L}_A$ is closed with respect to $+$ and $\cdot$.**
A **solution** of this equation/system $y' = Ay$ is a function $y \in C^1(\mathbb{R},\mathbb{R}^n)$ such that $y'(t) = Ay(t)$ for all t where $y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$.

We give a geometric interpretation of a solution. Let $y' = Ay$ be a system defined by an $n \times n$-matrix A. At each point $y \in \mathbb{R}^n$ there is a vector $Ay \in \mathbb{R}^n$. We thus obtain a *vector field* $F : \mathbb{R}^n \to \mathbb{R}^n, y \mapsto Ay$. At each point y we think the vector Ay pinned.

A solution of the system is a differentiable function $y : \mathbb{R} \to \mathbb{R}^n$ with $t \mapsto y(t)$, which fulfils the equation $y'(t) = Ay(t)$ for all t. Such a solution geometrically represents a curve in $\mathbb{R}^n$: For $n = 2$ it is a plane curve $t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and a space curve $t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$ for $n = 3$.

If we interpret t as time, the velocity $y'(t)$ along the curve at each point $y(t) \in \mathbb{R}^n$ is the vector $Ay(t) \in \mathbb{R}^n$. We can determine the direction $y'(t_0)$ of a point $y(t_0)$ on the solution curve at any time t_0. The solution curve is therefore tangential to the vector field at every point.



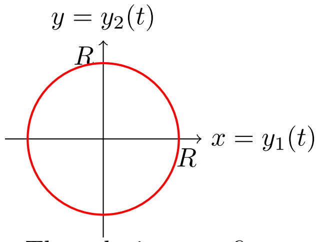
In the above figure a (red) plane solution curve is drawn and in two of its points the (blue) tangent vectors.

Let the vector field F be, for example, the velocity field of a fluid. If a particle is thrown into the flow at the time $t = 0$ at the location y_0, it is then carried along by the flow: At each point $y(t)$ of its journey $t \mapsto y(t)$ it therefore has the speed $Ay(t)$ given by the flow there. The journey of the particle realises exactly the solution of the initial value problem $y' = Ay, y(0) = y_0$. It is located at the time t at $y(t)$ given by the solution.

Exercise

Consider the system $\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$ as initial values.

Verify that the solution $y : \mathbb{R} \to \mathbb{R}^2$ is a circle. What is the flow direction?



Solve the system by finding solutions of the two ODEs. The solutions confirm the geometric observation.

1.2 Linear maps

Definition. *Let V and W be vector spaces.*

1. *A map $\mathcal{F} : V \to W, x \mapsto \mathcal{F}(x)$ is called* **linear map,** *if for all $x, y \in V$ and for all $\alpha \in \mathbb{R}$ we have*
   - *$\mathcal{F}(x + y) = \mathcal{F}(x) + \mathcal{F}(y)$ and $\mathcal{F}(\alpha x) = \alpha\mathcal{F}(x)$.*
2. *A* **bijective** *linear map is called* **isomorphism**. *The VS V and W are called* **isomorphic**.

Examples of linear Matrix-Vector-Product

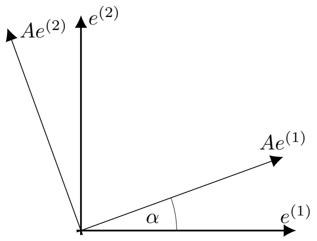
- $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-2y \\ 3x \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is linear.
- In general $A \in M_{m\times n}$ defines $\mathcal{F} : V = \mathbb{R}^n \to \mathbb{W} = R^m, x \mapsto Ax$.

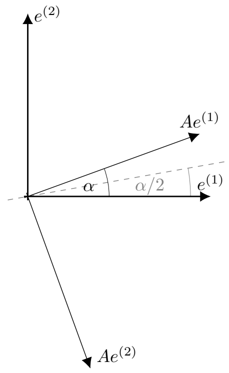
  Why is this linear?

  For $n = m$ it is an isomorphism $\iff$ matrix A is invertible.
- Rotation and Reflection

  $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ $\quad$ $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$

  Rotation with angle α $\quad$ Reflection at axis with slope $\frac{\alpha}{2}$





- The map $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ orthogonal projection onto xy-plane. In general a linear map $P : V \to V$ with $P \circ P = P$ is called **projection**.

Exercises

1. Which of the following maps are linear?
   (a) **Translation:** $\mathbb{R}^n \to \mathbb{R}^n, x \mapsto x + a$, for $0 \neq a \in \mathbb{R}^n$

(b) **Derivative:** For $V = C^1(]a,b[), W = C^0(]a,b[)$,

$$\mathcal{F} : C^1(]a,b[) \to C^0(]a,b[), f \mapsto \frac{df}{dx} = f'$$

(c) **Sampling:** Let $a \leq a_1 < \ldots < a_k \leq b$ and

$$\mathcal{F} : C^0([a,b]) \to \mathbb{R}^k,\ f \mapsto \begin{pmatrix} f(a_1) \\ \vdots \\ f(a_k) \end{pmatrix}$$

2. Let $U(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$. What is the geometric interpretation of $v \mapsto U(\alpha)v$?

1.3 Coordinates and Change of basis

Spanning sets For given vectors $v_1, \ldots, v_n \in V$ and numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we get a vector $\alpha_1 v_1 + \ldots + \alpha_n v_n$ called **linear combination** (LC) of v_i. The sum of two LC $\alpha_1 v_1 + \ldots + \alpha_n v_n$ and $\beta_1 v_1 + \ldots + \beta_n v_n$ is again a LC of v_i. Also the multiplication of a LC by a scaler becomes a LC of the same vectors v_i. Thus we define

Definition. *Let V be a VS and $v_1, \ldots, v_n \in V$. The set of all LC of v_i*

$$U = \{\alpha_1 v_1 + \ldots + \alpha_n v_n \mid \alpha_i \in \mathbb{R}\} = \langle\{v_1, \ldots, v_n\}\rangle$$

is a subspace of V. We say U is **spanned** *or* **generated** *by $v_1, \ldots, v_n \in V$.*

Exercises

1. Let W be the set of all vectors of the form $\begin{pmatrix} 5b + 2c \\ b \\ c \end{pmatrix}$, where b and c are arbitrary. Find vectors u and v such that $W = \langle\{u, v\}\rangle$.

2. Let W be the set of all vectors of the form $\begin{pmatrix} s + 3t \\ s - t \\ 2s - t \\ 4t \end{pmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

3. For the next two tasks one might want to use a CAS.

(a) Show that $w = \begin{pmatrix} 9 \\ -4 \\ -4 \\ 7 \end{pmatrix}$ is in the subspace of $\mathbb{R}^4$ spanned by v_1, v_2, v_3,

where $v_1 = \begin{pmatrix} 8 \\ -4 \\ -3 \\ 9 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -4 \\ 3 \\ -2 \\ -8 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -7 \\ 6 \\ -5 \\ -18 \end{pmatrix}$.

(b) Determine if $y = \begin{pmatrix} -4 \\ -8 \\ 6 \\ -5 \end{pmatrix}$ is in the subspace of $\mathbb{R}^4$ spanned by the columns of $A = \begin{pmatrix} 3 & -5 & -9 \\ 8 & 7 & -6 \\ -5 & -8 & 3 \\ 2 & -2 & -9 \end{pmatrix}$.

Question: Let V be a vector space. Is there a generating set $\{v_1, \ldots, v_m\}$ of V and how can we find it? Well, of course we could take a subset of vectors that contains all element of V. But we want to get a minimal set in order to understand the form of the elements in V.

Linear independent Let recall notion of linear independent in $\mathbb{R}^m$. For each vector w we have to ensure that we can find numbers $\beta_1, \beta_2, \ldots, \beta_n$ such that the equation $w = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ is fulfilled.

Exercise Actually the equation $w = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ is a system of linear equation $w = B \cdot \beta$. How are B and β defined?

If we choose $w = 0$ we have the special case $0 = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n \in \mathbb{R}^m$.

Definition (Linear Independent in $\mathbb{R}^m$). *We call $v_1, v_2, \ldots, v_n$ linear independent if the trivial solution $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ is the only one of the linear system $B \cdot \beta = 0$.*

Therefore we try to solve a $m \times n$ - system of linear equations using tools from Linear Algebra like Gauss elimination. To decide at least whether a non trivial solution exists we can apply the criteria given by the determinant or the rank of B, i.e. if $m = n$ we have to see whether $\det(B) \neq 0$. In case of $m > n$ we compute the rank and check whether $\text{Rank}(B) = n$?

Definition (Linear Independent in abstract V). *We call $v_1, v_2, \ldots, v_n \in V$ linear independent if the trivial solution $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ is the only one of the equation $0 = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$.*

Note that this definition in the same manner as for $\mathbb{R}^m$, but we don't have tools at hand like Gauss, Determinante etc.

Exercise Decide whether $\{p_i\}$ are linear independent in $\mathcal{P}_{\leq 3}$.

1. $p_1(x) = 1 + x^2, p_2(x) = 1 - x^2$
2. $p_1(x) = 1, p_2(x) = 1 + x, p_3(x) = 1 - x$.
3. $p_1(x) = x + 1, p_2(x) = x - 1, p_3(x) = x^2 + 1, p_4(x) = x^2 - 1$
4. $p_1(x) = x^3 - 2, p_2(x) = 2x^2 + 1, p_3(x) = -x + 2, p_4(x) = 1$

Definition. *If $\mathcal{B}$ is a* **minimal** *spanning set of V, we call $\mathcal{B}$ a* **basis** *von V. ("Minimal" meaning, that there is no subset of $\mathcal{B}$ spanning V).*

Fact (**Criteria**). *The set* $\mathcal{B} = \{b_1, \ldots, b_n\}$ *is a basis of* V *if only if one of the equivalent condition is fulfilled:*

i. $\langle \mathcal{B} \rangle = V$ **and** *the vectors* $b_1, \ldots, b_n$ *are* **linear independent**.

ii. *For each* $v \in V$ *exist* **unique coefficients** $\alpha_i \in \mathbb{R}$ *s.t.* $v = \alpha_1 b_1 + \ldots + \alpha_n b_n$.

Note Every VS has at least one basis.

Examples of bases

1. The column vectors of the unity matrix in $M_{n\times n}$ are a basis of $\mathbb{R}^n$, the standard basis e_i.
2. Each set of n linear independent vectors in $\mathbb{R}^n$ is a basis.
3. Monomials $1, x, x^2, \ldots, x^n$ form a basis of $\mathcal{P}_{\leq n}$.
4. **Application to solution space** $\mathcal{L}_A$

   Let $A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{13}{10} \end{pmatrix}$. The functions $t \mapsto \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$ and $t \mapsto \begin{pmatrix} 4e^{\frac{1}{2}t} \\ 3e^{\frac{1}{2}t} \end{pmatrix}$ form a basis of $\mathcal{L}_A \subset C^1(\mathbb{R}, \mathbb{R}^2)$ for the system $y' = Ay$.
5. **For Fourier in Part II**

   The functions $c_0, c_1, s_1, \ldots, c_N, s_N$ defined by

   $$c_0(x) = \frac{1}{\sqrt{2\pi}}, \;\; c_k(x) = \frac{1}{\sqrt{\pi}} \cos(kx), \;\; s_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx),$$

   are linear independent and therefore a basis of a subspace $T_N \subset C^0([a, b])$.

Exercises

1. Discuss and decide if $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$ form a basis.
2. Determine a basis of $\{A \in M_{2\times 2} | A^T = A\}$.
3. Let $x + 1, x - 1, x^2 + 1, x^2 - 1 \in \mathcal{P}_{\leq 2}$. Decide whether they generate $\mathcal{P}_{\leq 2}$, are linear independent or even form a basis. Likewise for

   $$\left\{ U_1(x) = x^3 - 2, U_2(x) = 2x^2 + 1, U_3(x) = 1, U_4(x) = -x + 2 \right\} \subseteq \mathcal{P}_{\leq 3}.$$
4. Find a basis for the set of vectors

   (a) in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

   (b) $\begin{pmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \\ 4 \\ -7 \\ 10 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 11 \\ -8 \\ -7 \end{pmatrix}$ (Use a CAS)

Fact (Dimension). *Let V be generated by a finite number of vectors. Each basis has the same number of n vectors and n is called dimension of V.*

Notation: $n = \dim V = \dim_{\mathbb{R}} V$.

Each set of n linear independent vectors in a VS of dimension n is a basis.

Examples

1. $\dim_{\mathbb{R}} \mathbb{R}^n = n = \dim_{\mathbb{C}} \mathbb{C}^n = n$ but $\dim_{\mathbb{R}} \mathbb{C}^n = 2n$.
2. $\dim_{\mathbb{R}} M_{n\times n} = n^2$ (only real entries) Q: What about symmetric matrices?
3. $\dim \mathcal{P}_{\leq n} = n + 1$
4. Let $\mathcal{K} = \{x|Ax = 0\}$ and $\mathcal{I}_A = \{b|Ax = b, \text{ for some } x\}$ be the subspaces above. There's a relation between the dimension of these subspaces:

   $$\boxed{n = \dim \mathcal{K}_A + \dim \mathcal{I}_A} \qquad \textbf{The Rank Theorem}$$

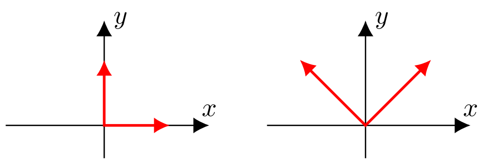
   The dimension $\dim \mathcal{I}_A$ is called the rank of the matrix.
5. The subspace $T_N \subset C^0([a, b])$ has dimension $2N + 1$.
6. $\dim C^n(\mathbb{R}) = \infty \leftarrow$ **dimension can be infinity.**

Coordinates and coordinate vector Let $\dim V = n$. With a choice of a basis $B = \{b_1, \ldots, b_n\}$ we identify V with $\mathbb{R}^n$. There is an isomorphism

$$\varphi_B : V \xrightarrow{\simeq} \mathbb{R}^n, \ v = \sum_{i=1}^{n} \alpha_i b_i \mapsto \varphi_{\mathcal{B}}(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = [v]_{\mathcal{B}}.$$

The $\alpha_i \in \mathbb{R}$ are unique determined by v, but **depend on the choice of a basis.** These are the **coordinates** of the vector v with respect to this chosen base $\mathcal{B}$. They form **the coordinate vector** $\varphi_{\mathcal{B}}(v) = \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$..

Example Let $n = 2$ and $v = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)$. If we choose the standard base $\mathcal{B} = \{e_1, e_2\}$ (see figure on the left)



$\mathcal{B} = \{e_1, e_2\}$ $\qquad$ $\mathcal{B}' = \{b'_1, b'_2\}$

we get $v = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) = 4\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + 2\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 4e_1 + 2e_2$. With respect to this base v has the coordinates 4 and 2, .i.e. the coordinate vector is $\varphi_{\mathcal{B}}(v) = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)$. Let $\mathcal{B}' = \{b'_1, b'_2\}$ be the basis with $b'_1 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $b'_2 = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ (see figure above on the right). Then we look for numbers α_1 and α_2 with $v = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) = \alpha_1 b'_1 + \alpha_2 b'_2 = \alpha_1 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + \alpha_2 \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$. The coordinate vector with respect to this base is $\varphi_{\mathcal{B}'}(v) = \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)$.

Exercises

1. Show that $\mathcal{B}' = \{b_1', b_2'\}$ in the above example is indeed a basis and solve the linear system above to determine $\varphi_{B'}(v)$.

2. Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \right\}$ form a basis of $\mathbb{R}^3$ and determine the coordinate vector $\varphi_{\mathcal{B}}(v)$ for $v = \begin{pmatrix} 5 \\ -1 \\ 9 \end{pmatrix}$.

3. In $\mathcal{P}_{\leq 3}$ we choose (standard-) basis 1, x, x^2, x^3.
   What is the coordinate vector $U_3(x) = 8x^3 - 4x$?

4. Find the vector x determined by the given coordinate vector $[x]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

   (a) $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right\}, \quad [x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

   (b) $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} \right\}, \quad [x]_{\mathcal{B}} = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$

5. Find the coordinate vector $[x]_{\mathcal{B}}$ of x relative to the basis given by b_i.

   (a) $b_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \quad x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

   (b) $b_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

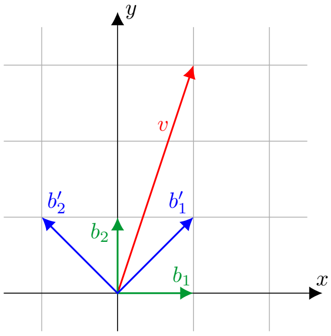
Change of Basis, Coordinate transformation Take two bases of the same n-dimensional VS $V^n = V$ denoted by $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{B}' = (b_1', \ldots, b_n')$. For a vector $v \in V$ there are two coordinate vectors: with respect to $\mathcal{B}$ and with respect to $\mathcal{B}'$.

$$[v]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \text{ and } [v]_{\mathcal{B}'} = \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} \qquad \boxed{\text{How are those related?}}$$

To answer this question start with $v = b_i$ compute the coordinate vector and form a matrix $T = ([b_1]_{\mathcal{B}'} \ldots [b_n]_{\mathcal{B}'})$. If we apply this to $v = b_i'$ we get another matrix $S = ([b_1']_{\mathcal{B}} \ldots [b_n']_{\mathcal{B}})$ and the relations $[v]_{\mathcal{B}'} = T[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}} = S[v]_{\mathcal{B}'}$.

Definition. *The matrix* $T = ([b_1]_{\mathcal{B}'} \ldots [b_n]_{\mathcal{B}'})$ *is the* **transformation matrix** *for the base change* $\mathcal{B} \rightsquigarrow \mathcal{B}'$. *For* $\mathcal{B}' \rightsquigarrow \mathcal{B}$ *the transformation is* $S = T^{-1}$.

Example Let $V = \mathbb{R}^2$ with $\mathcal{B} = (b_1, b_2) = (e_1, e_2)$ as standard basis and a new basis $\mathcal{B}' = (b'_1, b'_2) = (e_1 + e_2, e_2 - e_1)$.



It's $v = 2b'_1 + b'_2$ and

$$S = ([b'_1]_{\mathcal{B}}\, [b'_2]_{\mathcal{B}}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

We get

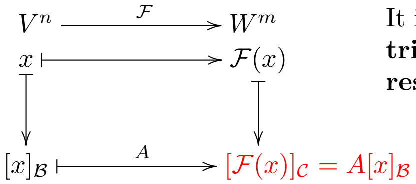
$$[v]_{\mathcal{B}} = S[v]_{\mathcal{B}'} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Exercises

1. Start with v in terms of the other basis.
2. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases for a vector space V, and suppose $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.
   (a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
   (b) Find $[x]_{\mathcal{C}}$ for $x = -3b_1 + 2b_2$. Use part (a).
3. Let $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{B} = \{b_1, b_2, b_3\}$ be bases for a vector space V, and suppose $a_1 = 4b_1 - b_2$, $a_2 = -b_1 + b_2 + b_3$, and $a_3 = b_2 - 2b_3$.
   (a) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.
   (b) Find $[x]_{\mathcal{B}}$ for $x = 3a_1 + 4a_2 + a_3$.

1.4 What is the matrix of a linear map?

We have the following situation with two vector spaces $V = V^n$ and $W = W^m$ of $\dim V = n$ and $\dim W = m$, a linear map $\mathcal{F} : V^n \to W^m, x \mapsto \mathcal{F}(x)$. We choose a basis of V denoted by $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_m\}$ a basis of W. For the vector $x \in V$ we get the coordinates $[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ and also for $y = \mathcal{F}(x) \in W$ the coordinates $[\mathcal{F}(x)]_{\mathcal{C}} = [y]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$. They sit in a diagram



How can we compute the matrix A? It is a $m \times n$-matrix and called **matrix of the linear map $\mathcal{F}$ with respect to $\mathcal{B}$ and $\mathcal{C}$.**

Each such linear map $\mathcal{F} : V^n \to W^m$ can be represented by an $m \times n$-matrix, depending on the choice of the bases of V and W. We apply the addtional steps to compute this representation matrix:

1. In the special case of a basis vector b_i compute $\mathcal{F}(b_i)$ for each i and afterwards the coordinates $[\mathcal{F}(b_i)]_\mathcal{C}$.

2. Form a matrix $\boxed{A = ([\mathcal{F}(b_1)]_\mathcal{C} \dots [\mathcal{F}(b_n)]_\mathcal{C})}$ i.e. the columns of A are the coordinate vectors of $\mathcal{F}(b_i)$ with respect to $\mathcal{C}$.

3. We get the formula $\boxed{[y]_\mathcal{C} = [\mathcal{F}(x)]_\mathcal{C} = A[x]_\mathcal{B}}$, i.e. to compute the coordinate vector of $y = \mathcal{F}(x)$ we multiply the matrix A with the coordinate vector of x.

Example Let $V = \mathcal{P}_{\leq 2}$ with $\mathcal{B} = \{1, x, x^2\}$ and $W = \mathcal{P}_{\leq 1}$ with $\mathcal{C} = \{1, x\}$. The derivative $\mathcal{F} : V \to W, p \mapsto p'$ defines a linear map. Compute

$$\mathcal{F}(1) = \frac{d}{dx}1 = 0, \mathcal{F}(x) = \frac{d}{dx}x = 1, \mathcal{F}(x^2) = \frac{d}{dx}x^2 = 2x$$

and $[\mathcal{F}(1)]_\mathcal{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, [\mathcal{F}(x)]_\mathcal{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, [\mathcal{F}(x^2)]_\mathcal{C} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. These three vectors are the columns of A, thus the matrix is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Let us test this with $p(x) = a + bx + cx^2$. We get

$$[p(x)]_\mathcal{B} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } [p'(x)]_\mathcal{C} = [b + 2cx]_\mathcal{C} = \begin{pmatrix} b \\ 2c \end{pmatrix}.$$

Indeed $[p'(x)]_\mathcal{C} = A[p(x)]_\mathcal{B}$:

$$\underbrace{\begin{pmatrix} b \\ 2c \end{pmatrix}}_{[p'(x)]_\mathcal{C}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{[p(x)]_\mathcal{B}}$$

Composition Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ bases in V^n, W^m and Z^p.

Let matrix A representing $\mathcal{F} : V \to W$ w.r.t. $\mathcal{B}$ and $\mathcal{C}$ and B the map $\mathcal{G} : W \to Z$ w.r.t. $\mathcal{C}$ und $\mathcal{D}$. We get a diagram

$$\begin{array}{ccccc} & & \mathcal{H} = \mathcal{G} \circ \mathcal{F} & & \\ V^n & \xrightarrow{\mathcal{F}} & W^m & \xrightarrow{\mathcal{G}} & Z^p \\ x & \longmapsto & \mathcal{F}(x) & \longmapsto & \mathcal{G}(\mathcal{F}(x)) = \mathcal{G} \circ \mathcal{F}(x) \\ [x]_\mathcal{B} & \longmapsto & A[x]_\mathcal{B} & \longmapsto & BA[x]_\mathcal{B} \end{array}$$

In particular with $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$, $\mathcal{G} : \mathbb{R}^m \to \mathbb{R}^p, y \mapsto By$ it follows $\mathcal{H} = \mathcal{G} \circ \mathcal{F} : \mathbb{R}^n \to \mathbb{R}^p, x \mapsto BAx$.

Change of basis revisited What happens to the matrix A representing a linear map $\mathcal{F} : V^n \to V^n$ if we change the basis?

Let $[\mathcal{F}]_{\mathcal{B}}$ and $[\mathcal{F}]_{\mathcal{B}'}$ representing $\mathcal{F}$ w.r.t. $\mathcal{B}$ and $\mathcal{B}'$.

Let T be coordinate change matrix $\mathcal{B} \rightsquigarrow \mathcal{B}'$. We get

$$\begin{array}{ccc} [v]_{\mathcal{B}} & \stackrel{\mathcal{F}}{\longmapsto} & [\mathcal{F}]_{\mathcal{B}}[v]_{\mathcal{B}} \\ \downarrow T & & \downarrow T \\ T[v]_{\mathcal{B}} = [v]_{\mathcal{B}'} & \stackrel{\mathcal{F}}{\longmapsto} & [\mathcal{F}]_{\mathcal{B}'}[v]_{\mathcal{B}'} = T[\mathcal{F}]_{\mathcal{B}}[v]_{\mathcal{B}} \\ & & \stackrel{*}{=} [\mathcal{F}]_{\mathcal{B}'}T[v]_{\mathcal{B}} \end{array}$$

If we multiply the equation $*$ with T^{-1} (from the right) we get a transformation rule $[\mathcal{F}]_{\mathcal{B}'} = T[\mathcal{F}]_{\mathcal{B}}T^{-1}$ given by the coordinate change matrix.

Fulfilling this relation the matrices $[\mathcal{F}]_{\mathcal{B}'}$ and $[\mathcal{F}]_{\mathcal{B}}$ are called to be conjugated.

Example Let $V = \mathbb{R}^2$, with bases $\mathcal{B}, \mathcal{B}'$ as in the above example 1.3 and consider $A = [\mathcal{F}]_{\mathcal{B}} = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$.

For the change $\mathcal{B}' \rightsquigarrow \mathcal{B}$ we had $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ therefore $T = S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Compute $[\mathcal{F}]_{\mathcal{B}'} = T[\mathcal{F}]_{\mathcal{B}}T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus with a clever choice of $\mathcal{B}'$ it might be possible to represent $\mathcal{F}$ by an easier matrix.

Exercises

1. In the last example compute A^n for $n = 2, 3, 4, \ldots$.
2. Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{D} = \{d_1, d_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that $T(b_1) = 3d_1 - 5d_2$, $T(b_2) = -d_1 + 6d_2$ and $T(b_3) = 4d_2$. Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$.
3. Let $V^3 = \mathbb{R}^3$ be with standard basis $\mathcal{B}$.

   A linear map $\mathcal{F} : V^3 \to V^3$ is defined by $A = \begin{pmatrix} -\frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

   (a) Choose a basis $\mathcal{B}' = \left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ to get new coordinates. Compute transformation matrix T for $\mathcal{B} \rightsquigarrow \mathcal{B}'$.

   (b) Compute B that represents $\mathcal{F}$ with respect to $\mathcal{B}'$.

   (c) What is the geometric meaning?

1.5 Diagonalisable

Definition. *A matrix $A \in M_{n\times n}(\mathbb{R})$ is called* **diagonalisable**, *if there exists an invertible T s.t.* $T^{-1}AT = J = D(\lambda_1, \lambda_2, ..., \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & & & 0 \\ & \lambda_2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & \lambda_{n-1} & 0 \\ & & & & \lambda_n \end{pmatrix}$.

Fact. *Diagonalisable is equivalent to that A has n linear independent eigenvectors $v_1, ..., v_n$, i.e. there exists an eigenbasis of $\mathbb{R}^n$. In this case the entries on the diagonal are the eigenvalues $\lambda_1, ...\lambda_n$ and the matrix T is $T = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$ i.e. the columns of T are eigenvalues of A.*

In the appendix we review basic facts on eigenvalues (EVal) and eigenvector (EVec).

Examples

1. If all EVal of a matrix A are simple i.e. $\lambda_i \neq \lambda_j$ if $i \neq j$, the matrix diagonalisable.
2. $A = \begin{pmatrix} 0 & -1 & 1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{pmatrix}$ with EVal -1,1,1 and EV $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$,$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$,$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Hence we get $T = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ that gives $T^{-1}AT = \text{diag}(-1, 1, 1) = J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
3. **Hermitian matrix (after Charles Hermite, 1822 - 1901)**

   Definition. *For a matrix $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ we define $\overline{A}^T = (\overline{a_{ji}})$, i.e. the entries are conjugated, further columns and rows are swapped.*

   In case of $A = \overline{A}^T$ it is called **hermitian**. *If all $a_{ij} \in \mathbb{R}$ and $A = A^T$ it is called* **symmetric**.

   Fact. *For such an A all n EVal of A are real and* **it is diagonalisable**.

Exercises

1. Which matrix is diagonalisable?

   (a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 0 & 0 & i \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 1 & 0 \\ -i & 0 & 0 & 0 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}$.

   (e) None.

2. Assume A diagonalisable. Does this imply that A is invertible? Or vice versa? Or both or no implications?

Chapter 2

Linear ODE-systems

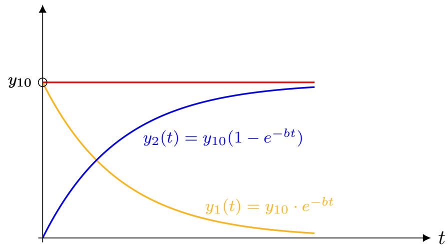
Let us start with an elementary example of two compartments $\boxed{K_1} \xrightarrow{b} \boxed{K_2}$ with $0 < b$. One might see this as an application, where K_1 is a source that supplies the use K_2. The mathematical model is given by the equation

$$y'(t) = Ay(t), \text{ where } y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \text{ and } A = \begin{pmatrix} -b & 0 \\ b & 0 \end{pmatrix}.$$

This equation is a collection of two ODE:

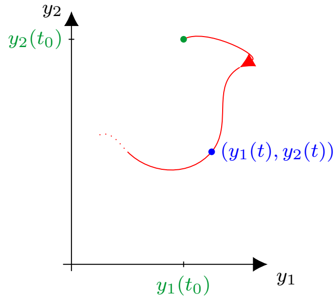
$$\begin{aligned} y_1'(t) &= -by_1(t) \\ y_2'(t) &= by_1(t) \end{aligned}$$

and we can find the solution directly by our knowledge from Calculus, given in the following coordinate system.

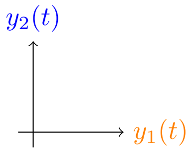


For each of the solution curves $t \mapsto y_1(t), t \mapsto y_2(t)$ we have chosen an initial value $y_1(t) = y_{10}$ and $y_2(t) = 0$.

Exercise Instead of plotting the solution curves $t \mapsto y_1(t), t \mapsto y_2(t)$ we can visualise a solution $t \mapsto \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ as a plane curve in a (y_1, y_2)-coordinate system. It would look like this example



How does this curve for the above example $\boxed{K_1} \xrightarrow{b} \boxed{K_2}$ look like?



Since we want to understand much more advanced systems, we won't be able to get solutions directly from Calculus. Instead we apply tools from Linear Algebra.

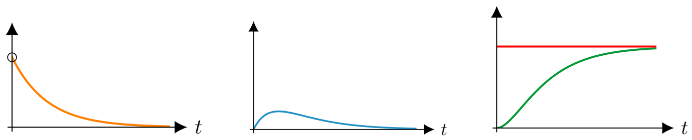
Hence where is the Linear Algebra? We can write the general solutions as a sum of vector

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-bt} \\ -C_1 e^{-bt} + C_2 \end{pmatrix} = C_1 e^{-bt} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{0t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of $A = \begin{pmatrix} -b & 0 \\ b & 0 \end{pmatrix}$ with eigenvalue $\lambda_1 = -b$ and eigenvalue $\lambda_2 = 0$.

Exercise Check this, i.e. verify, that in each case we get $Av_i = \lambda_i v_i$.

Yet another Example Consider a system with three compartments:
$\boxed{K_1} \xrightarrow{b_1} \boxed{K_2} \xrightarrow{b_2} \boxed{K_3}$ with $0 < b_1 < b_2$.
The ODEs are $y_1' = -b_1 y_1$, $y_2' = b_1 y_1 - b_2 y_2$ and $y_3' = b_2 y_2$.
Therefore (**check this** as an exercise) in the equation of the corresponding system of linear ODE $y'(t) = Ay(t)$ the matrix A is given by $A = \begin{pmatrix} -b_1 & 0 & 0 \\ b_1 & -b_2 & 0 \\ 0 & b_2 & 0 \end{pmatrix}$.
Solution curves look like this (compare the exercise below)

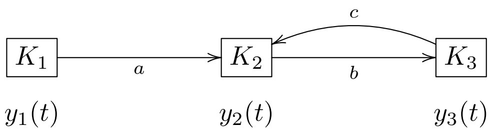


y_1 $\qquad$ y_2 $\qquad$ y_3

Exercises

1. Use Calculus to solve the above ODEs step by step:

   The first ODE $y_1' = -b_1 y_1$ is exponential decay and has been solved previously.
   The 2nd one $y_2' = b_1 y_1 - b_2 y_2$ is a bit more advanced, as one might need integrating factors.
   If one got y_2 one solves the last ODE $y_3' = b_2 y_2$ by plugging in and integration.

2. Let 

   Find a matrix A such that $y' = Ay$ models this system.

Again we are able to find the solutions by applying integration and other methods from Calculus. But in more complex situation we might run into difficult issues. Therefore Linear Algebra can offer an elegant way out. The main tools are eigenvalues (EVal) and eigenvectors (EVec).

2.1 Stationary Solutions of Linear ODE-System

Recall that $A \in M_{n\times n}$ defines $y' = Ay$, a **homogeneous** linear $n \times n$-system with $y'(t) = Ay(t)$ where $y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$.

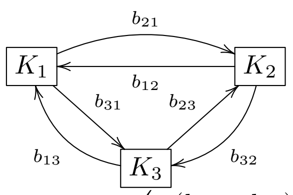
First, we will concentrate on the homogenous case, $y'(t) = Ay(t)$, i.e. there is no summand $g(t)$ on the right hand side of the equation. This means for a compartment system that there is no independent source interacting with the system. A solution is a map $y : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, such that for all t it is $y'(t) = Ay(t)$. A solution y_∞ with $y'_\infty = 0$ is called **stationary**. In case of a stationary solution the system is in equilibrium. Another key question is, whether such an equilibrium is stable. This means what happens to a solution y, that starts in the neighbourhood of y_∞. In the stable case, it will converge towards y_∞. There are several equivalent characterisations of a stationary solution coming from Linear Algebra:

- There exists always the trivial solution $y_\infty = 0$. If $\det(A) \neq 0$ it is unique.
- Stationary solutions are solutions y_∞ of the homogeneous system of linear equations $0 = A \cdot y_\infty$.
- If $\lambda = 0$ is an EVal and $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is an EVec of A the function $y : \mathbb{R} \to \mathbb{R}^n$ with $t \mapsto y(t) = e^{0t}v = v$ is stationary. Each EVec for EVal $\lambda = 0$ is such a solution. Those exists only if $\det(A) = 0$.

Exercises

1. Find stationary solutions $0 \neq y_\infty \in \mathbb{R}^3$ of $y' = Ay = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y$, with $0 \neq \omega \in \mathbb{R}$.

2. We have a system

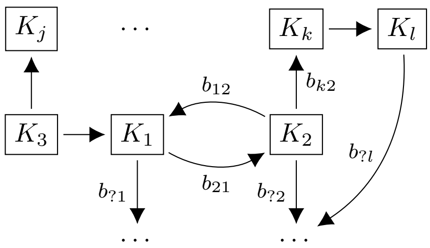


$$\begin{aligned} y_1' &= -(b_{31}+b_{21})y_1 + b_{12}y_2 + b_{13}y_3 \\ y_2' &= b_{21}y_1 - (b_{12}+b_{32})y_2 + b_{23}y_3 \\ y_3' &= b_{31}y_1 + b_{32}y_2 - (b_{13}+b_{23})y_3 \end{aligned}$$

with matrix $A = \begin{pmatrix} -(b_{31}+b_{21}) & b_{12} & b_{13} \\ b_{21} & -(b_{12}+b_{32}) & b_{23} \\ b_{31} & b_{32} & -(b_{13}+b_{23}) \end{pmatrix}$.

Show that such a system has always a stationary solution $0 \neq y_\infty \in \mathbb{R}^3$.

3. More general let



be a system. The corresponding equations are $y_i' = -\sum_{j=1}^{n} b_{ji}y_i + \sum_{j=1}^{n} b_{ij}y_j$.

They can be translated to $y' = Ay$ with

$$A = \begin{pmatrix} -\sum_{j=2}^{n} b_{j1} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & -\sum_{j=1,j\neq 2}^{n} b_{j2} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & -\sum_{j=1,j\neq 3}^{n} b_{j3} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & -\sum_{j=1}^{n-1} b_{jn} \end{pmatrix}.$$

Show that such a system has always a stationary solution $0 \neq y_\infty \in \mathbb{R}^n$.

2.2 Application to solution space $\mathcal{L}_A$

Fact (Basis solution). *Let λ be an EVal with EVec $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ of A.*

Then the function $y : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto y(t) = e^{\lambda t}v = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} x_1e^{\lambda t} \\ x_2e^{\lambda t} \\ \vdots \\ x_ne^{\lambda t} \end{pmatrix}$ is a solution of $y' = Ay$.

Exercises

1. Check the above facts by computing the derivative in each component and by using the eigenproperty $Av = \lambda v$.

2. A particle moving in a planar force field has a position vector x that satisfies $x' = Ax$. The 2×2-matrix A has eigenvalues 4 and 2, with corresponding eigenvectors $v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find the position of the particle at time t, assuming that $x(0) = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$.

Fact (Dimension of $\mathcal{L}_A$). *Consider $y' = Ay$ with matrix $A \in M_{n\times n}$. The solution space $\mathcal{L}_A \subset C^1(\mathbb{R}, \mathbb{R}^n)$ is an n-dimensional subspace. Hence to get a basis of $\mathcal{L}_A$ we need n linear independent vectors.*

For example those n linear independent vectors could be n linear independent EVec. Thus we are in the case of an eigenbasis, a basis formed by EVec.

Fact (Case of an eigenbasis). *Let A with EVal $\lambda_1, \ldots, \lambda_n$ delivering an eigenbasis $v_1, \ldots, v_n$ of $\mathbb{R}^n$. Then $\{t \mapsto e^{\lambda_1 t} v_1, \ldots, t \mapsto e^{\lambda_n t} v_n\}$ form a basis von $\mathcal{L}_A$. Each solution y can be written*

$$t \mapsto y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \ldots + C_n e^{\lambda_n t} v_n \text{ with constants } C_1, \ldots, C_n$$

Problem Not every $A \in M_{n\times n}$ provides an eigenbasis! We are going to meet a (new) tool: $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ for a quadratic matrix. This enables an algorithm to solve a system $y' = Ay$ in the general case.

Exercises

1. Find the general solution of $y' = Ay$ for $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $A = \begin{pmatrix} -2 & -5 \\ 1 & 4 \end{pmatrix}$.

2. Let $y' = Ay$ with $A = \frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$ and basis

$$\mathcal{B} = \left\{ t \mapsto e^{\lambda_1 \cdot t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, t \mapsto e^{-\frac{1}{3} \cdot t} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, t \mapsto e^{\lambda_3 \cdot t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

of $\mathcal{L}_A$. Determine λ_1, λ_3 and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

3. Use the above recipe to get the solution for $\boxed{K_1} \xrightarrow{b_1} \boxed{K_2} \xrightarrow{b_2} \boxed{K_3}$ and compare it with results coming from Calculus.

Hint: Use e.g. $v_1 = \begin{pmatrix} b_1 - b_2 \\ -b_1 \\ b_2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as EVec.

Classification in case $n = 2$ Let $y' = Ay$ with $A \in M_{2\times 2}$. Then we have the following cases

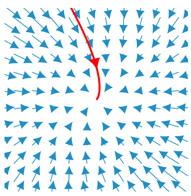
1. Let λ_1 and λ_2 be real EVal with EVec $v_1, v_2 \in \mathbb{R}^2$ **defining an eigenbasis**.
   (a) If $\lambda_1 \neq \lambda_2$ this is always the case. Then the **general solution** is given by $y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$.
   (b) With a double EVal $\lambda_1 = \lambda_2 = \alpha$ this can be **but does not have to be** the case of an eigenbasis. Then the **general solution** is given by $y(t) = C_1 e^{\alpha t} v_1 + C_2 e^{\alpha t} v_2 = e^{\alpha t}(C_1 v_1 + C_2 v_2)$.
   (c) For a double EW $\lambda_1 = \lambda_2 = \alpha$, for which **there isn't an eigenbasis**, we use later different methods: either e^A or we transform the system into an 2nd order ODE (Appendix).
2. If $\lambda_{1,2} = \alpha \pm i\beta \notin \mathbb{R}$ are EVal with EVec $v_1, v_2 \in \mathbb{C}^2$, these are always linearly independent, and we have as **general solution**

$$\begin{aligned} y(t) &= C_1 e^{(\alpha + i\beta)t} v_1 + C_2 e^{(\alpha - i\beta)t} v_2 = C_1 e^{\alpha t} e^{i\beta t} v_1 + C_2 e^{\alpha t} e^{-i\beta t} v_2 \\ &= e^{\alpha t}\left(C_1 e^{i\beta t} v_1 + C_2 e^{-i\beta t} v_2\right). \end{aligned}$$

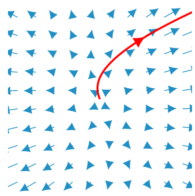
   To get a real representation in the applications the complex vector $y(t)$ can be rewritten $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}(K_1 \cos(\beta t) + K_2 \sin(\beta t)) \\ e^{\alpha t}(L_1 \cos(\beta t) + L_2 \sin(\beta t)) \end{pmatrix}$, with constants $K_{1,2}$ and $L_{1,2}$. We use here the Euler formulae from the complex numbers, for example: $\cos(t) = \frac{e^{it} + e^{-it}}{2}$.

With the EVal we can now understand the qualitative behaviour of a solution.

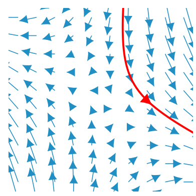
1. EVal λ_1, λ_2 real



$\lambda_1, \lambda_2 < 0$: "stable"

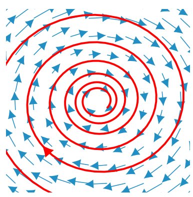


$\lambda_1, \lambda_2 > 0$: "instable"

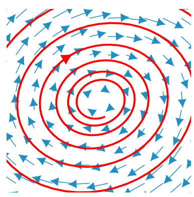


$\lambda_1 < 0 < \lambda_2$: "saddle point"

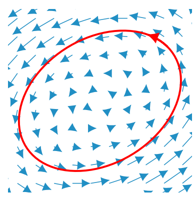
2. Complex EVal $\lambda_2 = \overline{\lambda_1}$



real part < 0: "stable"



real part > 0: "instable"



real part $= 0$: "center"

Exercise Find for each of the six cases an example of a matrix A, that fits. It should have real entries and at most one is $= 0$.

2.3 Exponential of a matrix

Consider $A \in M_{n\times n}(\mathbb{R})$ defining a system $y' = Ay$. We want to tackle the general question how to determine a basis of $\mathcal{L}_A \subset C^1(\mathbb{R},\mathbb{R}^n)$ if A is not diagonalisable (or we cannot decide on this)?

Let us recall from Calculus how e^x is defined for a real number x. It is given by the Taylor series:

$$\mathbb{R} \ni x \mapsto e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{k!}x^k + \ldots \in \mathbb{R}$$

We formally translate this into higher dimensions and obtain for $A \in M_{n\times n}$ a new matrix $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \in M_{n\times n}$.

Let us collect some properties of this construction:

1. Each summand $\frac{1}{k!}A^k$ is for $k = 0, 1, 2, 3, \ldots$ the matrix power A^k multiplied by the number $\frac{1}{k!}$, with $k! = 1 \cdot 2 \cdot \ldots \cdot (k-1) \cdot k$.
2. For the zero matrix $A = 0$ it is $e^0 = E_n$ ithe unit matrix, since all other summands are the zero matrix.
3. For every $A \in M_{n\times n}$ the series e^A converges. We do not want to specify here what we mean by convergence.
4. Note that $e^A = (\eta_{ij})$ is again an $n \times n$-matrix.
5. Let A be a function $A : \mathbb{R} \to M_{m\times n},\ t \mapsto A(t) = (a_{ij}(t))$ with functions as entries $a_{ij} : \mathbb{R} \to \mathbb{R}, t \mapsto a_{ij}(t)$. If $a_{ij} \in C^1(\mathbb{R})$, then the map A is differentiable with $A'(t) = (a'_{ij}(t))$.

   For $n = 1$ we get $A : \mathbb{R} \to M_{m\times 1}$, i.e. $A \in C^1(\mathbb{R},\mathbb{R}^m)$.

Fact (Main Application). *Let $A \in M_{n\times n}(\mathbb{R})$.*

1. *The function $\mathbb{R} \to M_{n\times n}$,*

   $$t \mapsto e^{tA} = E_n + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \ldots + \frac{1}{k!}t^kA^k + \ldots$$

   is differentiable with derivative $\left(e^{tA}\right)' = Ae^{tA}$.
2. *The column vectors of the matrix e^{tA} form a basis of the solution space $\mathcal{L}_A$.*
3. *As in the case $n = 1$ applies*
   (a) *The general solution y of $y' = Ay$ is written as*

   $$y(t) = \underbrace{e^{tA}C}_{\text{matrix times vector}} \in \mathbb{R}^n \text{ with } C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \quad C_i \text{ constant.}$$

   (b) *Let $y_0 \in \mathbb{R}^n$. Then the function y with $y(t) = e^{tA}y_0$ is the unique solution of the linear ODE-system $y' = Ay$ with $y(0) = y_0$.*

Question / Problem Apart from technical difficulties, we therefore have a well-defined procedure for determining a basis of $\mathcal{L}_A \subset C^1(\mathbb{R}, \mathbb{R}^n)$, but how do we calculate the exponential e^{tA} without the computer?

2.4 Methods to compute e^A

We start with matrices with a structure that allows us to compute the exponential using the definition and elementary manipulations.

Warm up Let $A = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$. What is $e^A =$? Hint: Compute A^2, A^3 and plug the powers into the definition of e^A.

Exponential of a diagonal matrix The exponential of a diagonal matrix is also a diagonal matrix.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \rightsquigarrow e^A = \begin{pmatrix} e^{a_{11}} & 0 & \dots & \dots & 0 \\ 0 & e^{a_{22}} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & e^{a_{nn}} \end{pmatrix}.$$

For this one applies that the powers of a diagonal matrix are diagonal again with powers of the diagonal entries on the diagonal. Again use the definition as a series.

Exponential of a block diagonal matrix A **block diagonal matrix** looks like this $A = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_k} \end{pmatrix}$, each $\boxed{A_i}$ is a quadratic matrix and they sit along the diagonal. All other entries are 0. For such A we get

$$A^j = \begin{pmatrix} \boxed{A_1^j} & & & \\ & \boxed{A_2^j} & & \\ & & \ddots & \\ & & & \boxed{A_k^j} \end{pmatrix} \rightsquigarrow e^A = \begin{pmatrix} \boxed{e^{A_1}} & & & \\ & \boxed{e^{A_2}} & & \\ & & \ddots & \\ & & & \boxed{e^{A_k}} \end{pmatrix}$$

Commutativity In general the product of two matrices A and B depends on the order, i.e. $AB \neq BA$.

Fact. *For $A, B \in M_{n\times n}$ there is a well-known rule*

$$\boxed{AB = BA} \implies e^{A+B} = e^A e^B \qquad \textbf{But not in general}$$

Application

1. Let us apply this to compute e^A with $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$. First we decompose A as $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{=B} + \underbrace{\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}}_{=C}$. Since $BC = CB$ we can apply the power rule $e^A = e^{B+C} = e^B e^C$. As a diagonal matrix $e^B = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix}$ and further
$$\begin{aligned} e^C &= E_2 + C + \frac{1}{2}C^2 + \ldots = E_2 + C + \frac{1}{2}\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}^2 + \ldots \\ &= E_2 + C + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \text{(only zero matrices)} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$
Hence $e^A = e^{B+C} = e^B e^C = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & 3e^2 \\ 0 & e^2 \end{pmatrix}$.

2. **Application for inverse** With $B = -A$ we see $A(-A) = (-A)A$ we get with the rule $e^A e^{-A} = e^{A+(-A)} = e^0 = E_n$ and therefore $\boxed{(e^A)^{-1} = e^{-A}}$.

Some background For the general case, we write e^{A+B} according to the definition as a power series. Then we use $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$. Even if the factors A and B do not commute, there are ways to calculate $e^A e^B$: For example, there is the *Baker-Campbell-Hausdorff formula*
$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{3!}[A,[A,B]]+\ldots} \text{ with } [A,B] = AB - BA.$$

H. F. Baker (1866 – 1956), J.E. Campbell (1862 – 1924) and Felix Hausdorff (1868 – 1942)

Exercises

1. Compute (with a CAS) e^{tA_i}, where $t \geq 0$ and A_i:

   (a) $A_1 = \begin{pmatrix} 7 & -2 \\ 0 & 7 \end{pmatrix}$

   (b) $A_2 = \begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}$

   (c) $A_4 = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$.

   (d) $A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$.

2. True or False?

   (a) $\det(e^A) \neq 0$.

   (b) $\det(e^A) = e^{\det A}$.

   (c) $\det(e^{-A}) = \left(\det\left(e^A\right)\right)^{-1}$

   (d) $\det\left(e^{A+B}\right) = \det\left(e^A\right)\det\left(e^B\right)$.

Eigenvalues and Determinante of e^A Let λ a EVal with EVec v of A. We compute with the definitions

$$
\begin{aligned}
\left(e^A\right) v &= \left(\sum_{k=1}^{\infty} \frac{1}{k!} A^k\right) v && \text{Definition } e^A \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \left(A^k v\right) && \text{Theory} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \left(\lambda^k v\right) && A^k \cdot v = \lambda^k \cdot v \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k!} \lambda^k\right) v && \text{Theory} \\
&\rightsquigarrow \boxed{\left(e^A\right) v = e^{\lambda} \cdot v}
\end{aligned}
$$

i.e. if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are EVal of A then $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$ are EVal of e^A.

Moreover, as $\det(A)$ is the product of the eigenvalues we have

$$\det\left(e^A\right) = e^{\lambda_1} \cdot e^{\lambda_2} \cdot \ldots \cdot e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \ldots + \lambda_n}.$$

In our strategy for calculating e^{tA}, we would like to transform the matrix A into a new matrix J, where e^{tJ} can be determined more easily. The theorem (Jordan normal form) below ensures that there is such a J for every A.

Firstly, we describe the structure of the matrix J and ensure that e^{tJ} can actually be specified directly.

Definition. *Let λ be a number.*

A matrix $J \in M_{n\times n}$ of the form $J_{\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \ldots & \\ 0 & \lambda & 1 & 0 & \ldots \\ & & \ddots & \ddots & \\ 0\ldots & & & \lambda & 1 \\ 0\ldots & & & & \lambda \end{pmatrix}$ *is called Jordan block of length n.*

In the following exercises we compute e^{tJ}.

Exercises

1. Start with case $\lambda = 0$ and compute the powers $J_0^2, J_0^3, \ldots$.

$$J_0 = B = \begin{pmatrix} 0 & 1 & 0 & \ldots & \\ 0 & 0 & 1 & 0 & \ldots \\ & & \ddots & \ddots & \\ 0\ldots & & & 0 & 1 \\ 0\ldots & & & & 0 \end{pmatrix} \rightsquigarrow B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0\ldots \\ & & \ddots & \ddots & \ddots \\ 0\ldots & & & 0 & 0 & 1 \\ 0\ldots & & & & 0 & 0 \\ 0\ldots & & & & & 0 \end{pmatrix} \ldots$$

and eventually $B^{n-1} = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \rightsquigarrow B^n = 0 = B^{n+1}$.

2. Decompose $J_\lambda = A + J_0$ with $J_0 = B$ and $AB = BA$ and use the commutativity $J_\lambda = J = \underbrace{\lambda E_n}_{=A} + \underbrace{\begin{pmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & 0 & \dots \\ & & \ddots & \ddots & \\ 0 & \dots & & 0 & 1 \\ 0 & \dots & & & 0 \end{pmatrix}}_{=B}$.

   We apply our power rule $e^{Jt} = e^{At+Bt} = e^{At}e^{Bt}$.

   By definition $\left(e^{\lambda t}E_n\right)\left(E_n + Bt + \frac{t^2}{2!}B^2 + \frac{t^3}{3!}B^3 + \dots + \frac{t^{n-1}}{(n-1)}!B^{n-1} + 0\right)$

   that composed to $e^{tJ} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \dots \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & & 1 & t & \frac{t^2}{2!} \\ 0 & \dots & & & 1 & t \\ 0 & \dots & & & & 1 \end{pmatrix}$.

Note that the entries on the minor diagonals are the coefficients of the Taylor series for e^t at $t_0 = 0$.

Application Conjugation The ransformation $A \rightsquigarrow J$ uses the so called conjugation: Two matrices A and J are conjugated if there is a $T \in M_{n\times n}$ with T invertible such that $T^{-1}AT = J$. If this is the case, we get

$$\boxed{e^A = T\ e^J\ T^{-1}} \quad \textbf{Cave!}\ \text{Order.}$$

Why is this so? With $T^{-1}AT = J$ we get $A = TJT^{-1}$ and we see

$$A^k = (TJT^{-1})^k = \underbrace{(TJT^{-1})(TJT^{-1})(TJT^{-1})\dots(TJT^{-1})}_{k \text{ factors}} = TJ^kT^{-1}$$

as the coloured factors cancel. Thus

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} = \sum_{k=0}^{\infty}\frac{TJ^kT^{-1}}{k!} = T\left(\sum_{k=0}^{\infty}\frac{J^k}{k!}\right)T^{-1} = Te^JT^{-1}.$$

Application to our purpose In order to solve $y' = Ay$ we need e^{tA}. If we know the matrix T we can compute e^{tJ} and afterwards $Te^{tJ}T^{-1}$.

Examples of conjugation

1. It's $A = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right)$ diagonalisable. Choose $T = \left(\begin{smallmatrix} 1 & -2 \\ 1 & 1 \end{smallmatrix}\right)$ with EVec as columns. With $T^{-1} = \frac{1}{3}\left(\begin{smallmatrix} 1 & 2 \\ -1 & 1 \end{smallmatrix}\right)$ we get $T^{-1}AT = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right) = D$ diagonal with exponential $e^{tD} = \left(\begin{smallmatrix} e^{5t} & 0 \\ 0 & e^{2t} \end{smallmatrix}\right)$ and $e^{tA} = Te^{tD}T^{-1} = \frac{1}{3}\left(\begin{smallmatrix} e^{5t}+2e^{2t} & 2e^{5t}-2e^{2t} \\ e^{5t}-e^{2t} & 2e^{5t}+e^{2t} \end{smallmatrix}\right)$. Compare this with outcome using the eigenbasis directly. It might look different, but with the choice of an initial values it becomes unique.

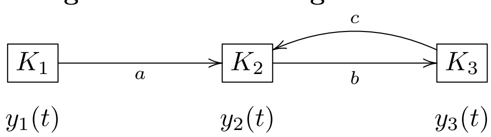
2. **Base change for $\mathcal{L}_A$** The general solution y of the equation $y' = Ay$ is given by $y(t) = e^{tA}C \in \mathbb{R}^n$, $C \in \mathbb{R}^n$.

   With conjugation it follows
$$e^{tA} = Te^{tJ}T^{-1} \rightsquigarrow y(t) = Te^{tJ}T^{-1}C = Te^{tJ}\left(T^{-1}C\right) = Te^{tJ}\widetilde{C}.$$
   This means that the general solution could be written in the form $Te^{tJ}\widetilde{C}$. Therefore the columns of Te^{tJ} form also a basis of $\mathcal{L}_A$, and again one might have some flexibility to find an appropriate basis, i.e. representation of the solutions.

Fact (**Jordan Normal Form Theorem (JNF)**).

1. *For each $A \in M_{n\times n}$ there exists a matrix $T \in M_{n\times n}$ s.t.*
$$T^{-1}AT = J = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix} \text{ with } J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & \\ 0 & \lambda_i & 1 & 0 & \dots \\ & & \ddots & \ddots & \\ 0 & \dots & & \lambda_i & 1 \\ 0 & \dots & & & \lambda_i \end{pmatrix} \in M_{n_i\times n_i}$$
   The matrix J is the Jordan normal form of A.
2. *The entries on the diagonal are the EVal λ_i of A.*
3. *If A is diagonalizable J is of the form $J = D(\lambda_1, \dots, \lambda_n)$.*

Example non diagonalisable vs. diagonalisable The model for

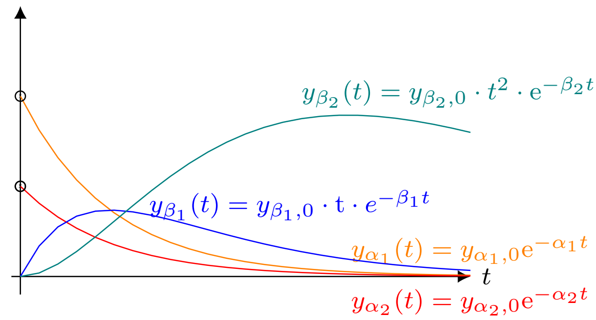


is the system $y' = Ay = \begin{pmatrix} -a & 0 & 0 \\ a & -b & c \\ 0 & b & -c \end{pmatrix} y$.

1. Choosing the parameters $a = \frac{2}{3}, b = \frac{1}{3}, c = \frac{1}{3}$ gives $A = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ with EVal $\lambda_{1,2} = -\frac{2}{3}$ (double) and $\lambda_3 = 0$. There is no eigenbasis as the EVec are of the form $v_{1,2} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. With $T = \begin{pmatrix} 0 & -3 & 0 \\ -1 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ we get the Jordan matrix $T^{-1}AT = \begin{pmatrix} -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} = J$. For initial value $y(0) = y_0$ the solution is therefore
$$y(t) = e^{At}y_0 = \left(Te^{tJ}T^{-1}\right)y_0 = \left(T \begin{pmatrix} e^{-\frac{2}{3}t} & te^{-\frac{2}{3}t} & 0 \\ 0 & e^{-\frac{2}{3}t} & 0 \\ 0 & 0 & 1 \end{pmatrix} T^{-1}\right) y_0.$$

2. If we choose the parameters $a = \frac{2}{3}, b = \frac{1}{3}, c = \frac{2}{3}$ it gives $A = \frac{1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$ with EVal $\lambda_{1,2,3} = -1, -\frac{2}{3}, 0$ and EVec $(v_1\ v_2\ v_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. In this case we have an eigenbasis.

Convergence: (non-)diagonalisable The difference whether A is diagonalisable or not is reflected by the qualitativ behaviour of the solution functions. In the case of diagonalisable the coordinate functions are given by terms of the form $t \mapsto e^{-\alpha t}$. On the other hand if A is non diagonalisable the coordinate functions look like $t \mapsto t \cdot e^{-\alpha t}$ or more general $t \mapsto q(t) \cdot e^{-\alpha t}$, where q is a polynomial in t.



Summary Let $A \in M_{n\times n}$. In order to solve $y' = Ay$, we try to follow the developed recipe:

1. Try to get (e.g. with the computer) T with $T^{-1}AT = J$. This is the JNF consisting of Jordan blocks $J_i = J = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$.

2. For each block $J_i = J$ compute $e^{tJ} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots \\ & & \ddots & \ddots & \ddots & \\ 0 & \ldots & & 1 & t & \frac{t^2}{2!} \\ 0 & \ldots & & & 1 & t \\ 0 & \ldots & & & & 1 \end{pmatrix}$

3. Compute for the block matrix

$$e^{tJ} = \begin{pmatrix} e^{tJ_1} & & & \\ & e^{tJ_2} & & 0 \\ & & \ddots & \\ & 0 & & e^{tJ_k} \end{pmatrix} \text{ mit } J = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix}$$

4. Eventually compute $e^{tA} = Te^{tJ}T^{-1}$. The column vectors form a basis of the solution space.

   Recall that the general solution y of $y' = Ay$ is $y(t) = e^{tA}C \in \mathbb{R}^n,\ C \in \mathbb{R}^n$. Using $e^{tA} = Te^{tJ}T^{-1}$ we get $y(t) = Te^{tJ}T^{-1}C = Te^{tJ}\left(T^{-1}C\right) = Te^{tJ}\widetilde{C}$ as a representation of the general solution, i.e. the columns of Te^{tJ} also define a basis.

Example/Exercises Conjugation and JNF

1. Compute (with a CAS) the JNF and the conjugate matrix T

   (a) $A = \begin{pmatrix} 1 & -3 & -2 \\ -1 & 1 & -1 \\ 2 & 4 & 5 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 0 & -1 & 1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{pmatrix}$

2. Verify and complete that with $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ that

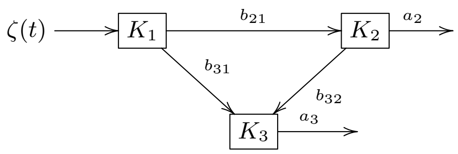
$$T^{-1}AT = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = J \rightsquigarrow e^{tJ} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$$\ldots e^{tA} = Te^{tJ}T^{-1}.$$

Chapter 3

Nonhomogeneous Case

Till now we had *homogeneous* systems of the form $y' = Ay$. Usually one encounters *nonhomogeneous* systems like $y' = Ay + g$. Let us look at the introductory example



with $A = \begin{pmatrix} -(b_{31}+b_{21}) & 0 & 0 \\ b_{21} & -(a_2+b_{32}) & 0 \\ b_{31} & b_{32} & -a_3 \end{pmatrix}$ and $g(t) = \begin{pmatrix} \zeta(t) \\ 0 \\ 0 \end{pmatrix}$.

Stationary solutions Elementary but again central solutions of $y' = Ay + g$ are stationary solutions, that represent an equilibrium of the system. A constant solution of the system is a **stationary solutions**. Such a stationary $y(t) = y_\infty$ can only exist if g is a constant function. It is a necessary condition that g is constant. Assume that there is a stationary $y_\infty \in \mathbb{R}^n$.

Hence $0 = y'_\infty = Ay_\infty + g(t) \rightsquigarrow g(t) = -Ay_\infty \in \mathbb{R}^n$. Thus $g(t) = g$ must be constant. If, in addition to g being constant, we also know that A is invertible, then we even can determine y_∞ by solving the equation $g = -Ay_\infty$ for y_∞ and it is $-A^{-1}g = y_\infty$.

If we write $0 = y'_\infty = Ay_\infty + g \iff Ay_\infty = -g$, we ask the question whether there is (exactly) one solution y_∞ or any number of solutions of a nonhomogeneous linear system of equations. We can decide this using Linear Algebra, i.e. with the determinant and/or the rank.

Exercises

1. Find stationary solutions y_∞ for $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $g(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{3} \end{pmatrix}$, $g(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. Let $y' = \begin{pmatrix} -3 & 0 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $y_\infty = \begin{pmatrix} 1 \\ y_{\infty,2} \end{pmatrix}$ stationary, what is $y_{\infty,2}$?

Principal methods to find solution For *nonhomogeneous* $y' = Ay + g$ there are there principal method to find solution:

1. Determining particular solution (as for ODE)
2. Decoupling in case of diagonalisable
3. Integrating Factors (as for ODE)

Method with particular solution Let $A \in M_{n\times n}$ and $g \in C^1(\mathbb{R}, \mathbb{R}^n)$ and we are looking for a nonhomogeneous solution $y \in C^1(\mathbb{R}, \mathbb{R}^n)$ of $y' = Ay + g$ (I). If $g = 0$ constant we get a **homogeneous** system $y' = Ay$ (H) and we know how to solve it. As in 1-dimensional case we have the recipe: The general solution of (I) is the sum of a particular solution of (I) and the general solution of (H).

The **general solution of (H)** is $y_H(t) = e^{tA}C$ that leads to the general solution for (I) with $y(t) = y_p(t) + e^{tA}C$.

Choosing an initial value for (H) we have $y_H(t) = e^{tA}y_0$, where $y_0 = y(0)$. Together it yields for (I): $\boxed{y(t) = y_p(t) + e^{tA}(y_0 - y_p(0))}$.

Examples/Exercise If possible, a stationary solution $y_p = y_\infty$ might be good choice. Take for example the system $y' = Ay + g$ above with $\zeta(t) = \zeta$ constant and $A = \begin{pmatrix} -(b_{31}+b_{21}) & 0 & 0 \\ b_{21} & -(a_2+b_{32}) & 0 \\ b_{31} & b_{32} & -a_3 \end{pmatrix}$, $g(t) = g = \begin{pmatrix} \zeta \\ 0 \\ 0 \end{pmatrix}$.

Check that $\det A \neq 0$ and therefore $y_\infty = -A^{-1}g$ and the solution with initial value $y(0) = y_0$ is $\boxed{y(t) = y_\infty + e^{tA}(y_0 - y_\infty)}$.

Exercises Apply the this method to the system $y' = Ay + g$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $g = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{3} \end{pmatrix}$, $g = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Method by decoupling in case of eigenbasis Let $g \in C^1(\mathbb{R}, \mathbb{R}^n)$ not necessarily constant. We again look for the solutions $y \in C^1(\mathbb{R}, \mathbb{R}^n)$ of the *nonhomogeneous* system $y' = Ay + g$ and assume that A **is diagonalisable**. There is therefore an invertible T with $T^{-1}AT = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$.

Let $x(t) = T^{-1}y(t)$, thus $y(t) = Tx(t)$. With the linearity of the matrix-vector product, the derivative is $y'(t) = (Tx(t))' = Tx'(t)$. Inserted into the system

$$
\begin{aligned}
y'(t) = Ay(t) + g(t) &\rightsquigarrow Tx'(t) = A(Tx(t)) + g(t) && \text{substitution} \\
&\rightsquigarrow T^{-1}(Tx'(t)) = T^{-1}((AT)x(t) + g(t)) && \text{from left } T^{-1}. \\
&\rightsquigarrow \underbrace{(T^{-1}T)}_{=E_n} x'(t) = \underbrace{(T^{-1}AT)}_{=D} x(t) + \underbrace{T^{-1}g(t)}_{=h(t)}
\end{aligned}
$$

We obtain a new nonhomogeneous system: $x'(t) = Dx(t) + h(t)$, that is written in coordinates: $\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$. Each coordinate gives an ODE, i.e. we have to solve n nonhomogeneous ODE

$$\begin{aligned} x_1' &= \lambda_1 x_1 + h_1 \\ x_2' &= \lambda_2 x_2 + h_2 \\ &\vdots \\ x_n' &= \lambda_n x_n + h_n. \end{aligned}$$

As these are decoupled it can be solved separately, for example by the method integrating factors, we get $x_i(t) = e^{\lambda_i t} \int e^{-\lambda_i s} h_i(s)\, ds + C_i e^{\lambda_i t}$. With $x = T^{-1}y$, the solution we are looking for is then given by $y = Tx$.

Example Apply this to $A = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$ and $g(t) = \begin{pmatrix} \zeta(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. We read the EVal from the diagonal and conclude that there is an eigenbasis. We choose $T = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow T^{-1}AT = \operatorname{diag}\left(-1, -\frac{1}{2}, -\frac{1}{3}\right)$.

Now we substitute $y = Tz \rightsquigarrow y' = Tz'$. In the original system $y' = Ay + g$ we get $Tz' = ATz + g$ and after multiplication form the left with T^{-1} it becomes decoupled $z' = T^{-1}ATz + T^{-1}g = \operatorname{diag}\left(-1, -\frac{1}{2}, -\frac{1}{3}\right) z + T^{-1}g$.

With $T^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ -1 & -1 & 0 \\ \frac{3}{2} & 1 & 1 \end{pmatrix}$ we get explicitly:

$$\begin{aligned} z_1' &= -1 z_1 - 1 \rightsquigarrow z_1(t) = c_1 e^{-t} - 1 \\ z_2' &= -\frac{1}{2} z_2 - 2 \rightsquigarrow z_2(t) = c_2 e^{-t/2} - 4 \\ z_3' &= -\frac{1}{3} z_3 + 3 \rightsquigarrow z_3(t) = c_3 e^{-t/3} + 9 \end{aligned}$$

Exercises

1. Get the above solution with re-substitution $y(t) = Tz(t)$.
2. Apply the first method of particular solution to solve the system and compare your solutions.
3. Find the solutions
   (a) $y'(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{3} \end{pmatrix} y(t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
   (b) $y'(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{3} \end{pmatrix} y(t) + \begin{pmatrix} -t \\ t \end{pmatrix}$ with $y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   (Linear ODE with non constant coefficients)

Higher dimensional Integrating Factors For a nonhomogeneous initial value problem $y' = Ay + g$ with $y(0) = y_0$ the **solution** can be written directly as

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-\tau)A}g(\tau)d\tau,$$

with integration in each coordinate.

To understand this, we make the approach for the homogeneous ODE system as in the one-dimensional case and start with $y(t) = e^{tA}C(t)$. Here, $y(t) = e^{tA}C$ is the general solution of the **homogeneous** system $y' = Ay$ and C after variation is a function $C \in C^1(\mathbb{R}, \mathbb{R}^n)$. To calculate $y`(t) = \left(e^{tA}C(t)\right)'$, we use:

i) Due to the linearity of the matrix-vector multiplication $e^{tA}C(t)$, the product rule also applies to matrix or vector-valued functions.

ii) It is $(e^{tA})' = Ae^{tA}$.

Thus follows $y'(t) = \left(e^{tA}C(t)\right)' \overset{\text{i)}}{=} (e^{tA})'C(t) + e^{tA}C'(t) \overset{\text{ii)}}{=} Ae^{tA}C(t) + e^{tA}C'(t)$.
The approach and the derivative are now used in $y`(t) = Ay(t) + g(t)$:

$$Ae^{tA}C(t) + e^{tA}C'(t) = Ae^{tA}C(t) + g(t).$$

Using $(e^{tA})^{-1} = e^{-tA})$ and $C'(t) = e^{-tA}g(t)$ we integrate in each coordinate

$$C(t) = \int_0^t e^{-uA}g(\tau)d\tau + \begin{pmatrix} K_1 \\ \vdots \\ K_n \end{pmatrix}$$

with n integration constants $K_1, ..., K_n \in \mathbb{R}$.
We insert this $C(t)$ into $y(t) = e^{tA}C(t)$, and the general solution of the nonhomogeneous ODE follows $y(t) = e^{tA}C(t) = e^{tA}\begin{pmatrix} K_1 \\ \vdots \\ K_n \end{pmatrix} + e^{tA}\int_0^t e^{-\tau A}g(\tau)d\tau$.
With $y(0) = y_0$ is $\int_0^{t=0} e^{-\tau A}g(\tau)d\tau = 0$ and thus must be $\begin{pmatrix} K_1 \\ \vdots \\ K_n \end{pmatrix} = y_0$.

Exercise Check that with this method we find the solution of the system

$$y'(t) = Ay(t) + g(t), \quad A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 1 \\ e^{-t} \end{pmatrix}, \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

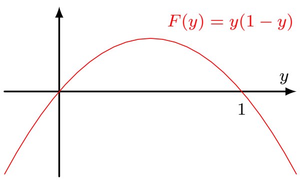
as $y(t) = \begin{pmatrix} \frac{1}{2}t^2e^{-t} + 2te^{-t} \\ \frac{1}{2}t^2e^{-t} + 3rd^{-t} + 2e^{-t} - 1 \end{pmatrix}$.

Chapter 4

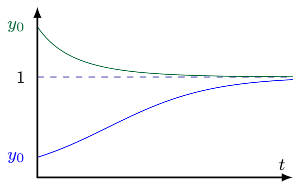
Linearisation

We describe a one-dimensional model using an ODE of the form $y'(t) = F(y(t))$ with initial value $y(0) = y_0$. We assume that we are able to write the ODE in an explicit form $y'(t) = F(y(t))$. The right-hand side is then an expression that depends just on y. Our system is an equilibrium y_∞ if $y'_\infty = 0$. What happens if we disturb the equilibrium y_∞? How do the solutions behave in the neighbourhood of an equilibrium solution y_∞? Do they converge towards y_∞? Can we specify a criterion for convergence near y_∞? Thus we are interested in more closer look at the behaviour near to a stationary solution. In the Appendix we give some foundational material on these topics.

Examples stationary solutions We start with the example $y' = y(1-y)$:

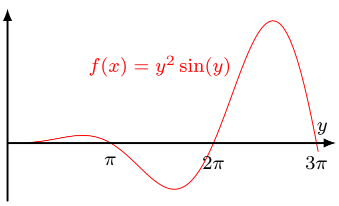


$y' = y(1-y)$
$y_{\infty,0} = 0, y_{\infty,2} = 1$

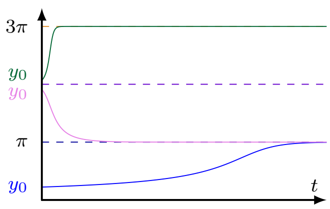


Solution with
$y(0) > 1$ and $y(0) < 1$

The stationary solution $y_{\infty,0}$ attracts solutions nearby and therefore we call it *attractor*. The other one $y_{\infty,1}$ repels solutions and we see it as a *repeller*.



$y' = y^2 \sin(y)$



A curve remains in $y_{\infty,k} = k \cdot \pi, k = 0, 1, 2, \ldots$
i.e. range $[k\pi, (k+1)\pi]$ where it started

Here we have just numerical solutions and we assume that the stationary solution $y_{\infty,k} = k\pi$ is a repeller for an even k and an attractor for an odd k.

Linearisation aka try to describe solution locally near y_∞ How can the behaviour of solutions nearby a stationary solution be determined more precisely?

We choose the Ansatz $y(t) = y_\infty + h(t)$ with a small perturbation $h(t)$ of a stationary solution y_∞ with $F(y_\infty)$ in the ODE.

$$y' = (y_\infty + h)' = y'_\infty + h' = 0 + h' = h' \overset{*}{=} F(y_\infty + h).$$

The Taylor approximation for the function $y \mapsto F(y)$ at the point y_∞ applies with $y = h + y_\infty$ and $h = y - y_\infty$

$$F(y_\infty + h) \overset{\text{Taylor}}{=} \underbrace{F(y_\infty)}_{=0} + F'(y_\infty)h + \text{ThO}.$$

The terms of higher order (ThO) contain powers $h^2, h^3, \ldots$. For a small h, the powers become even smaller, so that we consider the higher order terms (ThO) to be negligibly small. **Linearisation** now means that we omit the ThO.

With $h' \overset{*}{=} F(y_\infty + h)$ we get for h approximately the ODE $h' = F'(y_\infty)h$ with the solution $h(t) = h_0 e^{F'(y_\infty)t}$. Inserting this into the approach gives the approximate solution to the initial condition $y(0) = y_0$ or $h_0 = h(0) = y_0 - y_\infty$:

$$y(t) = y_\infty + (y_0 - y_\infty)e^{F'(y_\infty)t}$$

In summary, the following criterion results:

Fact. Linearisation $y' = F(y)$

1. *The stationary solutions of the nonlinear equation $y' = F(y)$ are the zeros of the right-hand side F.*

2. *The solution of the linearised equation $h' = F'(y_\infty)h$ behaves locally nearby the stationary solution y_∞ like the exact solution of $y' = F(y)$:*

   If $F'(y_\infty) < 0$, then $e^{F'(y_\infty)t} \to 0$ for $t \to \infty$. For solutions y that start close enough to y_∞, $y(t) \to y_\infty$ follows for $t \to \infty$.

   In this case, the stationary solution y_∞ is a attractor *or synonymously a* stable equilibrium*.*

   If $F'(y_\infty) > 0$, the following applies: No matter how close a solution $y \neq y_\infty$ starts at y_∞, the values $y(t)$ move away from y_∞.

   In this case, the stationary solution y_∞ is a repeller *or synonymously a* unstable equilibrium*.*

Exercise Apply this criterion in the example above $y' = y^2 \sin(y)$ to verify that $y_{\infty,k} = k\pi$ is a repeller for even k and an attractor for odd k.

Non linear models in higher dimension We focus on the case $n = 2$ and start with a *two-dimensional nonlinear ODE-system* $y' = F(y)$ where the right hand side is defined by a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{pmatrix}$ with two times continuously differentiable functions $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$. A solution of this equation $y' = F(y)$ is a function $t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.

As in the one-dimensional case, a stationary solution $y_\infty = \left(\begin{smallmatrix} y_{\infty,1} \\ y_{\infty,2} \end{smallmatrix}\right)$ is a zero of the right-hand side F, i.e. $y'_\infty = 0 = F(y_\infty) = \left(\begin{smallmatrix} F_1(y_{\infty,1}, y_{\infty,2}) \\ F_2(y_{\infty,1}, y_{\infty,2}) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$.

For the (in-)stability of y_∞, we distinguish between the two cases:

Stable case (attractor): Every solution y of a system $y' = F(y)$ that starts close enough to y_∞ converges against y_∞ for $t \to \infty$.

Unstable case (repeller): No matter how close a solution $y \neq y_\infty$ is to y_∞, the values $y(t)$ move away from y_∞ for $t \to \infty$.

Exercises Find stationary solutions $y' = F(y)$ in the four examples

1. $F \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_1^2 y_2 \\ 1 - y_2 + y_1 y_2 \end{pmatrix}$
2. $F \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_2 e^{y_1} + e \cdot y_2 \\ y_1 - y_2^3 \end{pmatrix}$
3. $F \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2^{y_1} y_2 - 2y_2 \\ y_1 - y_2^2 \end{pmatrix}$
4. $F \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a(y_1 y_2 - y_1^2) \\ \cos(y_1) - \sin(y_2) \end{pmatrix}$ where $a > 0$.

By linearising $y' = F(y)$ for the stationary solution y_∞ we try to describe the actual solution of the system and thereby make a decision about the stability. To do this, we first need the two-dimensional Taylor expansion:

Fact (Two-dimensional Taylor expansion)**.** *Let* $f : \mathbb{R}^2 \to \mathbb{R} \in C^2(\mathbb{R}^2, \mathbb{R})$*. Then*

$$f(x_1 + h_1, x_2 + h_2) = f(x_1, x_2) + \frac{\partial f(x_1, x_2)}{\partial x_1} h_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} h_2 + ThO.$$

For a suitable constant C *one estimates* $|ThO| < C(h_1^2 + h_2^2)$.

Again ThO denotes terms of higher order and thus summarises all expressions in h_1 and h_2 of the form $h_1^2, h_2^2, h_1 \cdot h_2, \ldots$.

As in the one-dimensional case for the linearisation we start with an Ansatz by a *perturbation* function h, i.e. $y(t) = y_\infty + h(t)$. In the ODE-system $y' = F(y)$ we get $y' = (y_\infty + h(t))' = \underbrace{y'_\infty}_{=0} + h' = F(y_\infty + h)$.

For the right-hand side $F(y_\infty + h)$ we apply Taylor to $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ and express this compactly as a matrix-vector product

$$h' = F(y_\infty + h) = \begin{pmatrix} F_1(y_{\infty,1} + h_1, y_{\infty,2} + h_2) \\ F_2(y_{\infty,1} + h_1, y_{\infty,2} + h_2) \end{pmatrix} = \ldots$$

$$\ldots = \underbrace{\begin{pmatrix} F_1(y_{\infty,1}, y_{\infty,2}) \\ F_2(y_{\infty,1}, y_{\infty,2}) \end{pmatrix}}_{=0} + \underbrace{\begin{pmatrix} \frac{\partial F_1(y_{\infty,1},y_{\infty,2})}{\partial y_1} & \frac{\partial F_1(y_{\infty,1},y_{\infty,2})}{\partial y_2} \\ \frac{\partial F_2(y_{\infty,1},y_{\infty,2})}{\partial y_1} & \frac{\partial F_2(y_{\infty,1},y_{\infty,2})}{\partial y_2} \end{pmatrix}}_{=DF(y_\infty)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \text{ThO}.$$

In the process of linearisation we abandon the ThO and get a **homogeneous linear** ODE-system $h' = DF(y_\infty)h$ and we know how solve such a linear system. The exponential of the matrix $DF(y_\infty)$ delivers the solution $h(t) = e^{tDF(y_\infty)}h_0$ where $h_0 = h(0)$ is the initial vector.

With $y = y_\infty + h \rightsquigarrow h = y - y_\infty$ and with $y(0) = y_0$ also $h_0 = y_0 - y_\infty$ and further $h(t) = e^{tDF(y_\infty)}h_0$

$$\rightsquigarrow y(t) - y_\infty = e^{tDF(y_\infty)}(y_0 - y_\infty) \rightsquigarrow y(t) = y_\infty + e^{tDF(y_\infty)}(y_0 - y_\infty)$$

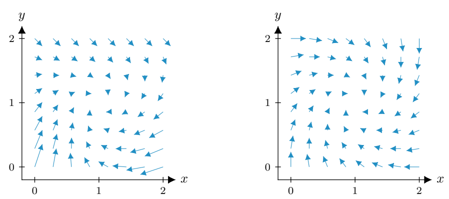
Let's summarise (This procedure works also for $F : \mathbb{R}^n \to \mathbb{R}^n$):

Let $y' = F(y)$ be a nonlinear ODE-system with vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$. To find an approximation of a solution $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, we can proceed as follows:

1. Calculate stationary solutions $y_\infty = \begin{pmatrix} y_{\infty,1} \\ y_{\infty,2} \end{pmatrix}$. If there are any, calculate $DF = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}$. The matrix is the Jacobian matrix, according to C. Jacobi (1804 – 1851)
2. Insert each y_∞ in the Jacobian matrix
$$DF(y_\infty) = \begin{pmatrix} \frac{\partial F_1(y_{\infty,1},y_{\infty,2})}{\partial y_1} & \frac{\partial F_1(y_{\infty,1},y_{\infty,2})}{\partial y_2} \\ \frac{\partial F_2(y_{\infty,1},y_{\infty,2})}{\partial y_1} & \frac{\partial F_2(y_{\infty,1},y_{\infty,2})}{\partial y_2} \end{pmatrix} \in M_{2\times 2}$$
3. The associated linear system provides an approximation solution. With initial $y(0) = y_0$ one gets $y(t) = y_\infty + e^{t\cdot DF(y_\infty)}(y_0 - y_\infty)$.

Example Let $y' = F(y)$ with $F : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} 1-2y_1+y_1y_2 \\ 3-2(y_1+y_2)+y_1y_2 \end{pmatrix}$.

For $y_\infty = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ as a fixed point the Jacobian matrix becomes $DF(1,1) = \left(\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ with $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$ as EVal. If we plot the vector fields we see a similar behauvior locally close to the fixed point.



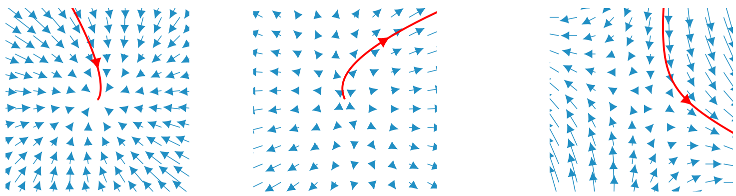
non linear Linear approximation

When does the linearisation describe the solution? With linearisation, we have a well-defined procedure to transform the solution of a **nonlinear system** $y' = F(y)$ into the solution of a **linear system.** Now we want to know whether and how this approximation provides information about the solution of $y' = F(y)$. To do this, we use the following result:

Fact (Hartman-Grobman[1](1959/60))**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ a vector field with fixed point $y_\infty = \binom{y_{\infty,1}}{y_{\infty,2}}$ and Jacobian matrix $DF(y_\infty)$. If each eigenvalue of $DF(y_\infty)$ has real part $\neq 0$, the solution of the linearised system approximates the true solution of the non-linear equation $y' = F(y)$ in a neighborhood of y_∞. The theorem also applies to $F : \mathbb{R}^n \to \mathbb{R}^n$.*

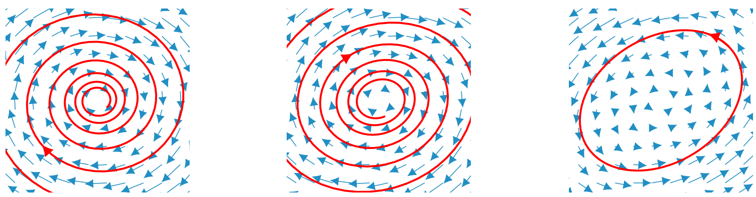
As an example, we consider the qualitative solution behaviour of a solution w.r.t. $y_\infty = \binom{0}{0}$ for the linearisation. Using the methods for linear models we had the the following classification given by the EVal. Without knowing the solution, that starts nearby y_∞, exactly, we can say something about its behaviour, applying the EVal of $DF(y_\infty)$.

1. EVal λ_1, λ_2 real



$\lambda_1, \lambda_2 < 0$: stable $\quad$ $\lambda_1, \lambda_2 > 0$: instable $\quad$ $\lambda_1 < 0 < \lambda_2$: saddle point

2. EVal $\lambda_1, \lambda_2 = \overline{\lambda_1}$



real part < 0: stable $\quad$ real part > 0: instable $\quad$ real part $= 0$: center

Note that the theorem does not work in the case of a center.

Exercises: Decide stability in the three cases of the above exercise:

2. $y_\infty = \begin{pmatrix} y_{\infty,1} \\ y_{\infty,2} \end{pmatrix}$ with $y_{\infty,1} > 0$ und $y_{\infty,2} > 0$.

3. $y_\infty = \begin{pmatrix} y_{\infty,1} \\ y_{\infty,2} \end{pmatrix}$ with $y_{\infty,1} > 0$ und $y_{\infty,2} > 0$.

4. $y_\infty = \begin{pmatrix} \pi/4 \\ \pi/4 \end{pmatrix}$.

[1] According to P. Hartman (1915 – 2015) and D. Grobman

Part II

Euclidean Spaces and Fourier Series

Chapter 5

Fourier via Integration

The idea of Fourier (Jean-Baptiste Joseph Fourier, 1768 – 1830) was: Try to express the values $f(x)$ of a 2π-periodic function as

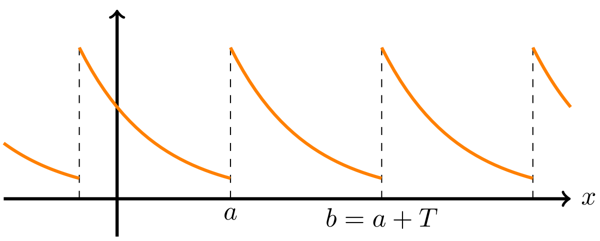
$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)).$$

We are not restricted to the case of period 2π and there are actually quite simple formulae for a_k, b_k and there are little assumptions on f.

5.1 Periodic Functions

Definition. *A function $f : \mathbb{R} \to \mathbb{R}$ is called* T**-periodic** *if for a positive T the values are $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. The smallest period of f is the* **prime periode**.

Given a function $f : [a, b[\to \mathbb{R}$ on an interval. This can be extended to a periodic extension $\widetilde{f} : \mathbb{R} \to \mathbb{R}$. It is common to denote $\widetilde{f}$ again with f, but usually only on $[a, b[$ one knows $\widetilde{f}(x) = f(x)$. One might adjust $\widetilde{f}(x)$.

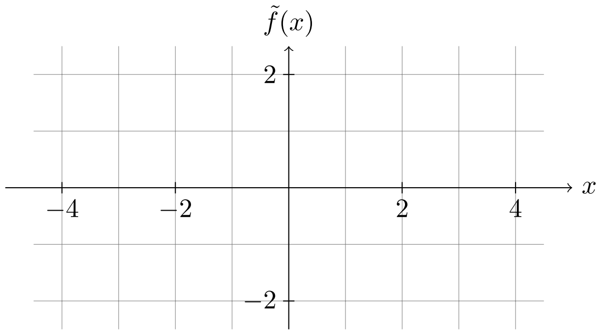


In this example we construct an T-periodic with $T = b - a$.

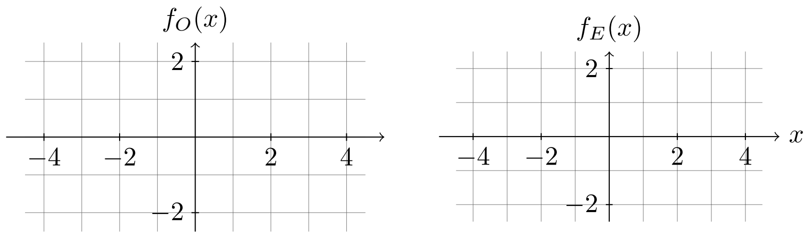
Exercises

1. Let $f : [-1, 1[\to \mathbb{R}$ a function with $f(x) = x$. Let $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ be the

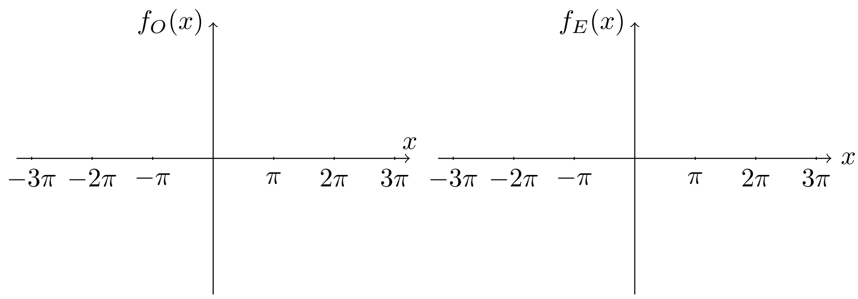
continuation with period 2. Sketch the graph of $\widetilde{f}$ in $]-3,3[$ and determine the term $\widetilde{f}(x)$ for $x \in [1,2[$.



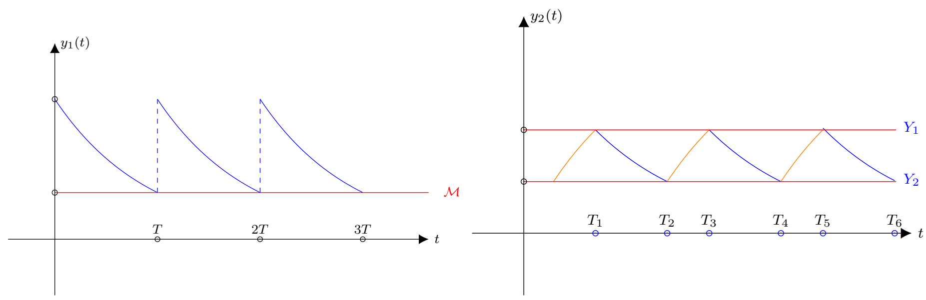
2. Let $f : [0,1[\to \mathbb{R}$ with $f(x) = 1 - x$. Let f_O be the **odd** continuation of f with period 2 and f_E be the **even** continuation of f with period 2. Sketch both graphs in $]-4,4[$.



3. Let $f : [0,\pi[\to \mathbb{R}$ with $f(x) = x^2$. Let f_O be the **odd** continuation of f with period 2π and f_E be the **even** continuation of f with period 2π. Sketch both graphs for $-3\pi < x < 3\pi$.



4. Let $y_1(t) = y_{1,0}e^{-bt}$ and $y_2(t) = y_{1,0}(1 - e^{-bt})$. Determine T and T_i, if the values $y_1(t)$ and $y_2(t)$ are bounded as indicated below.



Facts. *1. If $f_1, f_2, \dots, f_n$ are with period T any $\sum_{i=1}^{n} \alpha_i f_i$ has period T, also any (finite) products of f_i.*

2. By substitution we can change the period, ie. if f is T-periodic, the function g with $g(x) = f\left(\frac{Tx}{2\pi}\right)$ is 2π-periodic:

$$g(x + 2\pi) = f\left(\frac{T(x + 2\pi)}{2\pi}\right) = f\left(\frac{Tx}{2\pi} + T\right) = f\left(\frac{Tx}{2\pi}\right) = g(x)$$

Thus the developed theory for period 2π can be generalised to an T-periodic function.

5.2 Trigonometric Polynomial and Series

Definition. *A Function $\mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{a_0}{2} + \sum_{k=1}^{N}(a_k \cos(kx) + b_k \sin(kx))$ is called* **Trigonometric Polynomial** *of degree N - with period 2π. If we set $N = \infty$, we get* **Trigonometric Series** $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$.

If the series converges (e.g. for all x), the sum an 2π-periodic function.

With these trigonometric polynomials, the values $f(x)$ of a given function f can be approximated and, with the trigonometric series, they can even be represented exactly. We therefore examine the questions: How can we approximate periodic f, for which f und which x is $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$ and how to compute a_k and b_k?

Example Trigonometric Addition Theorem Without developing further theory we get representations like

$$\begin{aligned}
\cos^2(x) &= \frac{1}{2} + \frac{1}{2}\cos(2x) \\
\sin^3(x) &= \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x) \\
(\cos^2 \sin^3)(x) &= \frac{1}{8}\sin(x) + \frac{1}{16}\sin(3x) - \frac{1}{16}\sin(5x) \\
(\cos^3 \sin^2)(x) &= \frac{1}{8}\cos(x) - \frac{1}{16}\cos(3x) - \frac{1}{16}\cos(5x).
\end{aligned}$$

There is $f(x)$ on the left and a trigonometric polynomial on the right, with only two or three coefficients $a_k \neq 0$ or $b_k \neq 0$.

Exercise In the fourth example, $a_1 = \frac{1}{8}$ and $a_3 = a_5 = -\frac{1}{16}$. Determine the coefficients for the other examples too.

The equations above usually follow using the complex numbers (Euler's formulae) and the binomial relation $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. For Euler's formulae, we recall that the Taylor series (cp. Calculus) applies:

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \dots \\ i\sin(x) &= ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} \dots \end{aligned} \quad \Big| +$$

$$\cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} \dots = e^{ix}$$

From (I): $e^{ix} = \cos(x) + i\sin(x)$ and (II) $e^{-ix} = \overline{e^{-ix}} = \cos(x) - i\sin(x)$ follows by addition and subtraction $(I) \pm (II)$

$$\frac{e^{ix} + e^{-ix}}{2} = \cos(x) \quad \text{and} \quad \frac{e^{ix} - e^{-ix}}{2i} = \sin(x)$$

Example

$$\begin{aligned} \cos^2(x) &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{1}{4}\left(\left(e^{ix}\right)^2 + 2e^{ix} \cdot e^{-ix} + \left(e^{-ix}\right)^2\right) \\ &= \frac{1}{4}\left(2 + e^{i2x} + e^{-i2x}\right) \\ &= \frac{1}{2} + \frac{1}{4}\left(e^{i2x} + e^{-i2x}\right) = \frac{1}{2} + \frac{1}{4}\left(2\cos(2x)\right) \\ &= \frac{1}{2} + \frac{1}{2}\cos(2x) \end{aligned}$$

Thus $a_k = \dots$ and $b_k = \dots$

Search for Fourier coefficients with integration In order to get formulae for a_k and b_k we need some integration. Most significant is

Fact (**Trigonometric Orthogonality** relations)**.** *For* $n, k = 0, 1, 2, \dots$ *we have*

1. $$\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \begin{cases} 2\pi & n = k = 0 \\ \pi & n = k \neq 0 \\ 0 & n \neq k \end{cases}$$

2. $$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx = \begin{cases} 0 & n = k = 0 \\ \pi & n = k \neq 0 \\ 0 & n \neq k \end{cases}$$

3. $$\int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx = 0$$

The name will be explained later.

Exercise *Check one of these relations!*

Deducing Fourier coefficients via integration We assume that values are given by $f(x) = a_0/2 + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$. If we multiply on both sides with $\cos(nx)$ and calculate the integral, we get an expression for a_k and b_k.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx = &\int_{-\pi}^{\pi} \frac{a_0}{2} \cdot \cos(nx) dx \\ &+ \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx) dx + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin(kx)\cos(nx) dx \\ &= \int_{-\pi}^{\pi} \frac{a_0}{2} \cdot \cos(nx) dx + \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx) dx + 0.\end{aligned}$$

In the step to the second equal sign, the theory allows the sum and integral signs $\sum \leftrightarrow \int$ to be swapped. The red term in the third addend disappears due to the third orthogonal identity above. The following applies to the first summand:

$$\int_{-\pi}^{\pi} \frac{a_0}{2} \cdot \cos(nx) dx = \begin{cases} 0 & n \neq 0 \\ \displaystyle\int_{-\pi}^{\pi} \frac{a_0}{2} dx = \frac{a_0}{2}(2\pi) = a_0 \cdot \pi & n = 0 \end{cases}$$

For $n = 0, 1, 2, 3, \ldots$ only remains

$$\sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx) dx = \begin{cases} 0 & k \neq n \\ a_n \cdot \pi & k = n, \text{because of } \mathbf{1.} \text{ in Orthogonality} \end{cases}$$

Thus $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$ for every n.

If we multiply $f(x) = a_0/2 + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$ by $\sin(nx)$, we get the formulas for the b_k result due to the same reason. We summarise and define:

Definition. *Let f be a 2π-periodic function. The numbers*

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

are called Fourier coefficients of f. They define the Fourier series *of f*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)).$$

Exercise How do the formulae for a_k and b_k simplify in case of an odd or even f?

Use that sin is odd and cos is even and check that the product of two odd functions is even and the product of two even functions is even. Furthermore: The product of an odd and of an even function is odd. You should get that if f is even then $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) dx$ and $b_k = 0$. If f is odd, we get switched roles with $a_k = 0$ and $b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx$.

Exercises Compute a_k and b_k.

1. For the functions in Exercise 3. above.
2. For $f : [-\pi, \pi[\to \mathbb{R}$ with $f(x) = 3|x| - 2$ and 2π-periodic continuation.
3. For $f : [-\pi, \pi[\to \mathbb{R}$ with $f(x) = \sin^3(2x)$ and 2π-periodic continuation. Compare your results with the formulae you get using one of Euler's formula $\sin(\alpha) = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$ and $\sin^3(\alpha) = \dfrac{1}{8i^3}(e^{i\alpha} - e^{-i\alpha})^3 = \ldots$

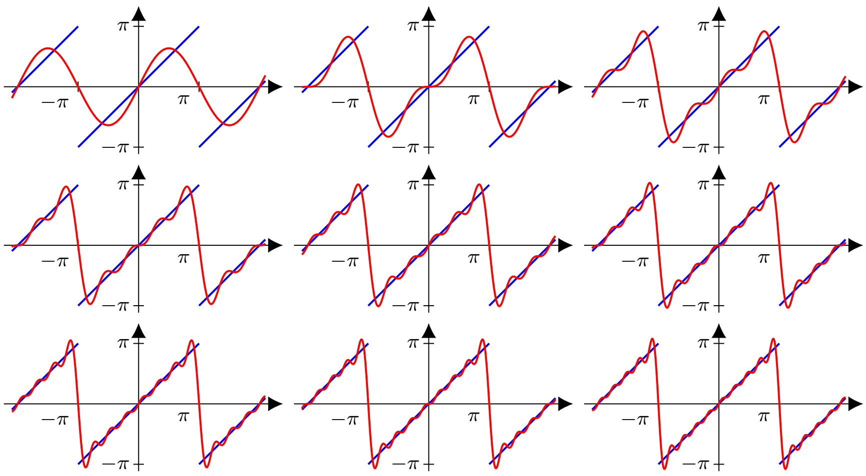
Pointwise convergence of Fourier series Since we got the formulae for the coefficients, we wonder for which f and x indeed the values are given by $f(x) = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$?

Theory provides us with an answer by defining a function $f : [-\pi, \pi] \to \mathbb{R}$ **piecewise continuous differentiable**, if f is continuous differentiable outside a finite number of points and in each exceptional or boundary point z exits the limits $\lim\limits_{\substack{x\to z\\x<z}} f(x) =: f(z^-)$, $\lim\limits_{\substack{x\to z\\x>z}} f(x) =: f(z^+)$, $\lim\limits_{\substack{x\to z\\x<z}} f'(x)$, $\lim\limits_{\substack{x\to z\\x>z}} f'(x) \in \mathbb{R}$

Fact (**Theorem by Dirichlet**[1]). *Let $f : [-\pi, \pi] \to \mathbb{R}$ be as above.*

- *Its Fourier series converges $\to f(x)$ for each continous point $x \in]-\pi, \pi[$.*
- *If f jumps at z the Fourier series converges towards $\dfrac{1}{2}(f(z^+) + f(z^-))$.*

Gibbs-Phenomenon Josiah Willard Gibbs, 1839 - 1903

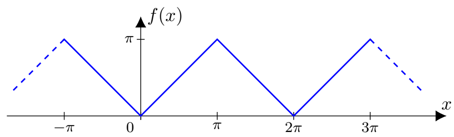


For each Fourier polynomial, the convergence at the jumps indicates the so-called Gibbs phenomenon. The small tower that forms to the left and right of the discontinuity is called Gibbs tower. It does not become smaller with increasing N, only narrower.

[1] Johann Peter Gustav Lejeune Dirichlet, 1805 - 1859

5.3 Solving ODE with Fourier

We apply the concept of Fourier to find a 2π-periodic solution of the 1. order ODE $y'(x) + y(x) = f(x)$ for



Ansatz and Comparison We make the Ansatz with assuming the a solution y can be represented by its Fourier series $y(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(kx) + \beta_k \sin(kx)$ and hence $y'(x) = \sum_{k=1}^{\infty} -k\alpha_k \sin(kx) + k\beta_k \cos(kx)$. This gives on the left side of the ODE:

$$y'(x) + y(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} (\alpha_k + k\beta_k) \cos(kx) + (\beta_k - k\alpha_k) \sin(kx)$$

$$\overset{\text{ODE}}{=} f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \text{ (Fourier for } f).$$

As soon we have a_k, b_k for f, we get the coefficients α_k, β_k for y via comparison. Since f is an even function the $b_k = 0$ and we get first $\beta_k - k\alpha_k = b_k = 0$ and then $\boxed{\beta_k = k\alpha_k}$.

Further $a_k = \alpha_k + k\beta_k = \alpha_k + k^2\alpha_k = \alpha_k(1 + k^2)$ and therefore $\boxed{\alpha_k = \frac{a_k}{1 + k^2}}$.

Particular solution By symmetrie of the even function f we compute a_k with $a_k = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx$. If $k = 0$ ist $a_0 = \pi$ and by the above $0 = \alpha_0$.

The coefficients with $k > 0$ need some Partial Integration (**P.I.**, see detail calculation below) $a_k = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx = \text{P.I.} = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{-4}{k^2\pi} & \text{if } k \text{ odd} \end{cases}$

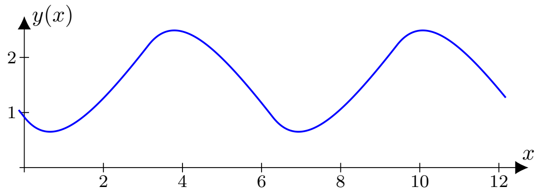
$$\begin{aligned} a_k &= \frac{2}{\pi} \int_0^{\pi} |x| \cos(kx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx \\ &= \frac{2}{\pi} \Big(\underbrace{x \frac{\sin(kx)}{k} \Big|_0^{\pi}}_{=0} - \int_0^{\pi} \frac{\sin(kx)}{k} dx \Big) \\ &= \frac{2}{\pi} \frac{\cos(kx)}{k^2} \Big|_0^{\pi} = \frac{2}{k^2\pi} (\cos(k\pi) - 1) \\ &= \begin{cases} 0 & \text{if } k > 0 \text{ even} \\ \frac{-4}{k^2\pi} & \text{if } k \text{ odd} \end{cases} \end{aligned}$$

Solution As $\boxed{\beta_k = k\alpha_k}$ and $\boxed{\alpha_k = \dfrac{a_k}{1+k^2}}$ we get

$$\alpha_k = \begin{cases} 0 & k \text{ even} \\ \frac{-4}{k^2(1+k^2)\pi} & k \text{ odd} \end{cases} \qquad \beta_k = \begin{cases} 0 & k \text{ even} \\ \frac{-4}{k(1+k^2)\pi} & k \text{ odd} \end{cases}$$

With odd indices $k = 2n+1$:

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2(1+(2n+1)^2)} + \frac{\sin((2n+1)x)}{(2n+1)(1+(2n+1)^2)}$$



Exercise/Example We can apply this also recipe for an 2nd order ODE: Let $4y''(x) + y(x) = f(x) = |x|$ for $x \in [-\pi, \pi]$ (ODE). We find the solution with the following steps

1. Solve the homogeneous equation $4y''(x) + y(x) = 0$.
2. Determine the coefficients of the 2π-periodic Fourier series of $|x|$ on $[-\pi, \pi[$. and find a particular solution y_p of ODE with the Ansatz for an even function $y_p(x) = \sum_{k=0}^{\infty} A_k \cos(kx)$.
3. Combine this to get the general solution of the ODE. Eventually find the solution (ODE) with initial values $y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right) = 0$.

Let's do this.

1. Write $4y''(x) + y(x) = 0 \rightsquigarrow y''(x) + \frac{1}{4}y(x) = 0$ to get with the characteristic equation $y_H(x) = C_1\cos(\frac{1}{2}x) + C_2\sin(\frac{1}{2}x)$.
2. The Fourier coefficients for the right hand side $f(x) = |x|$ are given in the above example by $b_k = 0, a_0 = \pi$ and $a_k = \begin{cases} 0 & \text{if } k > 0 \text{ even} \\ \frac{-4}{k^2\pi} & \text{if } k \text{ odd.} \end{cases}$

   The Ansatz $y_p(x) = \sum_{k=0}^{\infty} A_k \cos(kx)$ in (ODE) on the left hand side gives:

$$4\left(\sum_{k=0}^{\infty} A_k\cos(kx)\right)'' + \sum_{k=0}^{\infty} A_k\cos(kx) = 4\sum_{k=1}^{\infty} -k^2 A_k\cos(kx) + \sum_{k=0}^{\infty} A_k\cos(kx).$$

Then write both sides of (ODE) as Fourier series:

$$4\sum_{k=1}^{\infty} -k^2 A_k \cos(kx) + \sum_{k=0}^{\infty} A_k \cos(kx) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx).$$

Identify the Fourier coefficients of both sides in

$$4\sum_{k=1}^{\infty} -k^2 A_k \cos(kx) + \sum_{k=0}^{\infty} A_k \cos(kx) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx).$$

to get $A_k = \dfrac{a_k}{1-4k^2}$ for $k \neq 0$ and $A_0 = \dfrac{a_0}{2} = \dfrac{\pi}{2}$.

3. The general solution is therefore:

$$\begin{aligned} y(x) &= y_H(x) + y_P(x) \\ &= C_1 \cos\left(\frac{1}{2}x\right) + C_2 \sin\left(\frac{1}{2}x\right) + \frac{\pi}{2} + \sum_{k=0}^{\infty} A_{2k+1} \cos((2k+1)x). \end{aligned}$$

4. As $\cos\left((2k+1)\frac{\pi}{2}\right) = 0$ for all k the initial condition $y(-\frac{\pi}{2}) = y(\frac{\pi}{2}) = 0$ implies that $\frac{C_1}{\sqrt{2}} - \frac{C_2}{\sqrt{2}} + \frac{\pi}{2} = 0$ and $\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} + \frac{\pi}{2} = 0$. If you solve the linear system you get $C_1 = \dfrac{-\pi}{\sqrt{2}}$ and $C_2 = 0$ and eventually

$$\boxed{y(x) = \frac{-\pi}{\sqrt{2}} \cos\left(\frac{1}{2}x\right) + y_P(x).}$$

5.4 Generalisation

We know by Dirichlet's Theorem which 2π-periodic functions f and which x allow to write the values $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$ as values of the Fourier-series. And we compute the Fourier-coefficients a_k and b_k using the formulae $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx.$

What is the situation in the case of a period $T \neq 2\pi$?

Let $f : [-\frac{T}{2}, \frac{T}{2}] \to \mathbb{R}$ be a piecewise continuous differentiable function. By substitution we get a function g with $g(x) = f\left(\dfrac{Tx}{2\pi}\right)$ defined on $[-\pi, \pi]$. This has period 2π and therefore we use our knowledge to write this as a Fourier series

$$g(x) \overset{*}{=} \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

With a re-substitution $f(x) = g\left(\dfrac{2\pi x}{T}\right)$ this defines the T-periodic Fourier series of f by $\boxed{f(x) = \dfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\dfrac{2\pi x}{T}\right) + b_k \sin\left(k\dfrac{2\pi x}{T}\right)}$.

To get the coefficient a_k, b_k in this series we set $z = z(x) = \dfrac{2\pi x}{T}$ in

$$g(z) \overset{*}{=} \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kz) + b_k \sin(kz)$$

and apply substitution rules for integration

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(z)\cos(kz)dz = \frac{1}{\pi}\int_{-T/2}^{T/2} g\Big(\frac{2\pi x}{T}\Big)\cos\Big(k\frac{2\pi x}{T}\Big)\frac{2\pi}{T}dx \\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\Big(k\frac{2\pi x}{T}\Big)dx. \end{aligned}$$

If we use the same methods for the b_k, and we can summarise:

Fact (Fourier series with period T). *Let f be a T-periodic function. Its Fourier series is* $\frac{a_0}{2} + \sum_{k=0}^{\infty} a_k \cos\Big(k\frac{2\pi x}{T}\Big) + b_k \sin\Big(k\frac{2\pi x}{T}\Big)$ *where*

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\Big(k\frac{2\pi x}{T}\Big)dx \quad \textit{and} \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\Big(k\frac{2\pi x}{T}\Big)dx$$

Exercise Compute the coefficients a_k and b_k for the periodic functions in Exercises 1. and 2. above.

Complex Version There is also a complex version that enables us to give a more compact form that might be even offer a more efficient computation.

Let f be a function of the form $f : \mathbb{R} \to \mathbb{C}, t \mapsto f(t) = g(t) + ih(t) \in \mathbb{C}$. To integrate this we integrate real and imaginary part separately, i.e.

$$\int_a^b f(t)dt = \int_a^b g(t)dt + i\int_a^b h(t)dt.$$

With Euler's relations $\frac{e^{ix}+e^{-ix}}{2} = \cos(x)$, $\frac{e^{ix}-e^{-ix}}{2i} = \sin(x)$ and $\frac{1}{i} \overset{*}{=} -i$ we get a translation Real Fourier series $\longleftrightarrow$ Complex Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\Big(k\frac{2\pi x}{T}\Big) + b_k \sin\Big(k\frac{2\pi x}{T}\Big) \quad \longleftrightarrow \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=0}^{\infty} a_k\cos(kx) + b_k\sin(kx) && \text{Real Fourier series} \\ &= \frac{a_0}{2} + \sum_{k=0}^{\infty} \frac{a_k}{2}(e^{ikx} + e^{-ikx}) + \frac{b_k}{2i}(e^{ikx} - e^{-ikx}) && \text{Euler} \\ &= \frac{1}{2}\Big(a_0 + \sum_{k=1}^{\infty}(a_k - ib_k)e^{ikx} + \sum_{k=1}^{\infty}(a_k + ib_k)e^{-ikx}\Big) && \text{Minus sign, } \frac{1}{i} = -i \\ &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} && \text{Complex Version} \end{aligned}$$

$$\text{Hence } c_k = \begin{cases} \dfrac{a_k - ib_k}{2} & \text{if } k > 0 \\ \dfrac{a_0}{2} & \text{if } k = 0 \\ \dfrac{a_{-k} + ib_{-k}}{2} & \text{if } k < 0 \end{cases} \quad \text{resp.} \quad \begin{cases} a_k = c_k + c_{-k} \\ a_0 = 2c_0 \\ b_k = i(c_k - c_{-k}) \end{cases}$$

Closed Formula for complex coefficient In the above dictionary we need the real a_k and b_k to determine the complex c_k. There is also a direct way to get to the c_k. We discuss this in the example $T = 2\pi$ and $k > 0$:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos(kx) - i\sin(kx))dx \\ &= \frac{1}{2}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx - i\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx\Big) \\ &= \frac{1}{2}(a_k - ib_k) \rightsquigarrow \boxed{c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx} \end{aligned}$$

We summarise

Fact (Complex Fourier series). *Let $f : [-\frac{T}{2}, \frac{T}{2}[\to \mathbb{C}$. Then the complex Fourier series of f is*

$$\sum_{k=-\infty}^{\infty} c_k e^{2k\pi ix/T} \quad \textit{where} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)e^{-2k\pi ix/T}dx.$$

1. *For $f \in C^n(\mathbb{R})$ with period T converge the coefficients $c_k \to 0$ for $k \to \infty$ faster for n increasing.*
2. *In formula for c_k integration possible for any intervall of length T.*

Caveats Computation c_k In the above Formula for c_k the integration is possible for any interval of length T. **But be careful**: One must adjust the term $f(x)$ according to the choice of the interval.

Example Assume $f : \mathbb{R} \to \mathbb{R}$ is given as an T-periodic extension of

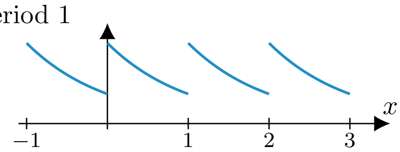
A) $[0, T[\to \mathbb{R}, x \mapsto g(x)$ or of B) $\left[-\frac{T}{2}, \frac{T}{2}\right[\to \mathbb{R}, x \mapsto h(x)$

The coefficient are computed by the above formula

A) $c_k = \frac{1}{T}\int_0^T g(x)\exp(\dots)\,dx$ B) $c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(x)\exp(\dots)\,dx$. But don't mix, e.g. $c_k = \frac{1}{T}\int_0^T h(x)\exp\left(-k\frac{2\pi i}{T}x\right)dx$

Exercises

1. Let $f(x) = e^{-x}$ für $0 \le x < 1$ with period 1



We get

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_0^T \exp(-x)\exp\left(-k\frac{2\pi i}{T}x\right)dx = \int_0^1 \exp\left(-(1+2k\pi i)x\right)dx \\
&= -\frac{1}{1+2k\pi i}\exp\left(-(1+2k\pi i)x\right)\Big|_0^1 = -\frac{1}{1+2k\pi i}\left(\exp\left(-1-2k\pi i\right)-1\right) \\
&= \frac{1-1/e}{1+2k\pi i} \rightsquigarrow a_k = c_k + c_{-k} = \frac{1-1/e}{1+2k\pi i} + \frac{1-1/e}{1-2k\pi i} = \frac{2-2/e}{1+4k^2\pi^2}
\end{aligned}
$$

Compute a_k, b_k for this f via real formulae.

2. Compute a_k, b_k and c_k for
   (a) the periodic functions above.
   (b) g with $g(t) = 3^{-t}, t \in [-1, 1[$ and 2-periodic continuation.

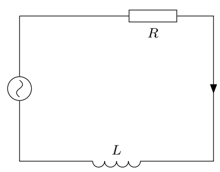
Exercise: Electric circuit with an external signal

1. Compute the Fourier coefficients for the T-periodic continuation of h with

$$
h(t) = \begin{cases} 4\cos\left(\frac{2\pi t}{T}\right), & \text{for } t \in [-\frac{T}{4}, \frac{T}{4}] \\ 0, & \text{for } t \in [-\frac{T}{2}, -\frac{T}{4}[\cup]\frac{T}{4}, \frac{T}{2}[\end{cases}, \qquad T > 0.
$$

2. Let h be the function above.

   The ODE $LI'(t) + RI(t) = h(t)$ models an electric circuit with an external signal by h.



   Find a particular solution and analyse the general solution for $t \to \infty$.

   Note that $h(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n i t}{T}}$ and use the Ansatz $\widetilde{I}_n(t) = A_n e^{2\pi i n t/T}$ to solve $L\widetilde{I}_n'(t) + R\widetilde{I}_n(t) = c_n e^{\frac{2\pi n i t}{T}}$.

Chapter 6

Fourier in LA disguise

Let $C^0([-\pi,\pi])$ be the vector space of continuous functions on $[-\pi,\pi]$: A trigonometric polynomial $\frac{a_0}{2}+\sum_{k=1}^{N}(a_k\cos(kx)+b_k\sin(kx))$ is linear combination of functions c_0, c_k and s_k with $c_0(x)=\frac{1}{\sqrt{2\pi}}$, $c_k(x)=\frac{1}{\sqrt{\pi}}\cos(kx)$ and $s_k(x)=\frac{1}{\sqrt{\pi}}\sin(kx)$ where $k=1,2,\ldots,N$. The scaling is for later purposes and will be revealed below.

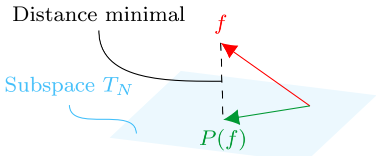
Consider the subspace $T_N=\langle\{c_0,c_1,s_1,c_2,s_2\ldots,c_N,s_N\}\rangle\subset C^0([-\pi,\pi])$. If we have a $f\in T_N$, then there are α_n and β_m s.t.

$$f(x)=\sum_{n=0}^{N}\alpha_n c_n+\sum_{m=1}^{N}\beta_m s_m=\underbrace{\frac{\alpha_0}{\sqrt{2\pi}}+\sum_{k=1}^{N}\left(\frac{\alpha_k}{\sqrt{\pi}}\cos(kx)+\frac{\beta_k}{\sqrt{\pi}}\sin(kx)\right)}_{\text{Trigonometric polynomial}}.$$

6.1 Euclidean Spaces

What is the connection of a_k, b_k and α_k, β_k and the Fourier coefficients?

And what do we do if $f\notin T_N\subset C^0([-\pi,\pi])$? To understand this, we need more theory. The idea is to try to find the best approximation of f by a function $P(f)\in T_N$ in the subspace T_N.

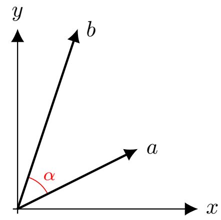


We'll see that the distance $f-P(f)$ minimal, if $P(f)$ is the **orthogonal projection** of f onto T_N and that the approximation improves for increasing N. For such a construction we need the **notion of orthogonal and distance** in an abstract VS, e.g. $C^0([-\pi,\pi])$.

Recap Euclidean $\mathbb{R}^n$ In $\mathbb{R}^2$ we have the standard scalar product $\cdot$ for two vectors $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ defined by $a \cdot b = a_1 b_1 + a_2 b_2 = |a||b|\cos(\alpha)$ where $|a|$ is the norm $|a| = \sqrt{a \cdot a}$.

Also $a \perp b \iff a \cdot b = 0$

Example: $\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} y \\ -x \end{pmatrix}$



The natural generalisation of the case $n = 2$ is given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Note There are other possibilities to define a scalar product on $\mathbb{R}^n$.
For example

1. For $n = 2$ define $\langle x, y \rangle = 2x_1 y_1 + 3x_2 y_2$.
2. With $A \in M_{n \times n}$ invertible define $\langle x, y \rangle = (Ax)^T Ay$.

Question What is an abstract scalar product? What do the constructions have in common?

Definition (Scalar Product (SCP) on a VS V). *It is a map*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, \ (a, b) \mapsto \langle a, b \rangle$$

such that for all $a, b, c \in V$ and $\lambda \in \mathbb{R}$ the following holds

Symmetry $\langle a, b \rangle = \langle b, a \rangle$

Bilinear $\langle a, b + c \rangle = \langle a, b \rangle + \langle a, c \rangle$
$\langle a, \lambda b \rangle = \lambda \langle a, b \rangle$ *Due to symmetry also in 1st component*

Positive definite $\langle a, a \rangle \geq 0$ *and* $\langle a, a \rangle = 0$ *only if* $a = 0$

A **Euclidean VS** $(V, \langle \cdot, \cdot \rangle)$ *is a VS V with scalar product $\langle \cdot, \cdot \rangle$.*
Two vectors $a, b \in V$ are **orthogonal** $\iff \langle a, b \rangle = 0$.

Examples/Exercises On the vector space of functions $V = C^0([a, b])$ we define a SCP by $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.

1. Check that this is indeed a SCP.

2. Let $a = -\pi$ and $b = \pi$ and $c_k, s_k \in C^0([-\pi, \pi])$ with

$$c_0(x) = \frac{1}{\sqrt{2\pi}}, \ c_k(x) = \frac{1}{\sqrt{\pi}} \cos(kx), \ s_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$$

Check that they fulfil the following equation

i. $\langle c_n, c_n \rangle = 1$ und $\langle c_n, c_k \rangle = 0$

ii. $\langle s_n, s_n \rangle = 1$ und $\langle s_n, s_k \rangle = 0$

iii. $\langle c_k, s_n \rangle = 0$

Hence 1. $c_k \perp c_n$, 2. $s_k \perp s_n$, if $k \neq n$ and 3. $c_k \perp s_n$. Use

i. $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \begin{cases} 2\pi & n = k = 0 \\ \pi & n = k \neq 0 \\ 0 & n \neq k \end{cases}$

ii. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx = \begin{cases} 0 & n = k = 0 \\ \pi & n = k \neq 0 \\ 0 & n \neq k \end{cases}$

iii. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx = 0$

3. Consider SCP $\langle f, g \rangle = \displaystyle\int_{-1}^{1} f(x)g(x)\,dx$. True or False?

(a) Curve of function f where $f(x) = x$ and of g where $g(x) = -x$ intersect orthogonal at zero, hence they are orthogonal with respect to SCP.

(b) Let f be an odd and g an even function, hence they are orthogonal with respect to SCP.

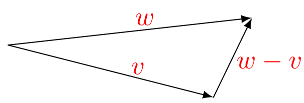
4. Let $V = C[0, 1]$, with SCP $\langle f, g \rangle = \displaystyle\int_{0}^{1} f(x)g(x)\,dx$.

(a) Compute $\langle f, g \rangle$, where $f(t) = 1 - 3t^2$ and $g(t) = t - t^3$.

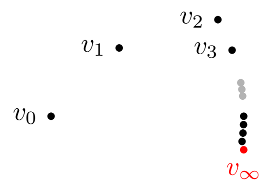
(b) Compute $\langle f, g \rangle$, where $f(t) = 5t - 3$ and $g(t) = t^3 - t^2$.

6.2 Normed spaces

For two vectors $v, w \in V$ with $V = \mathbb{R}^2$ or $V = \mathbb{R}^3$ our mathematical intuition and Pythagoras tell us that the distance is $|w - v|$, where $|v| = \sqrt{\langle v, v \rangle}$.



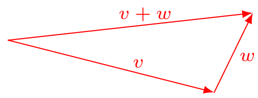
Morevover this leads to a notion of convergence: A sequence $(v_n)_{n\in\mathbb{N}}$ converges to $v \in V$, if $\lim_{n\to\infty} \|v_n - v\| = 0$.



We want to get this concept for a general vector space.

Definition. *1. A* **norm** *on a real VS V is a map* $\|\cdot\| : V \to \mathbb{R},\ v \mapsto \|v\|$ *such that for all* $v, w \in V$ *and* $\lambda \in \mathbb{R}$

- $\|v\| \geq 0$ *and* $\|v\| = 0 \iff v = 0$
- $\|\lambda v\| = |\lambda|\|v\|$
- $\|v + w\| \leq \|v\| + \|w\|$ *($\Delta - \neq$ Triangle inequality)*



A VS with **norm** *is a* **normed VS** $(V, \|\cdot\|)$

2. A vector $v \in V$ *with* $\|v\| = 1$ *is an* **unit vector.**

3. **Induced Norm** *The SCP induces a norm given by* $\|v\| = \sqrt{\langle v, v\rangle}$.

Examples

1. **Euclidean norm** on $\mathbb{R}^n$ ist given by $\|v\| = \sqrt{\sum_{k=1}^{n} v_k^2} = \sqrt{\langle v, v\rangle}$.

2. **On our favourite** VS $C^0([a, b])$ with SCP $\langle f, g\rangle = \int_a^b f(x)g(x)dx$ we get norm $\|f\|_{L^2} = \sqrt{\int_a^b f^2(x)dx}$ and unit vectors $c_0, c_1, s_1, \ldots, c_N, s_N$ with

$$c_0(x) = \frac{1}{\sqrt{2\pi}},\ c_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx),\ s_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx).$$

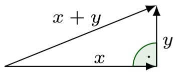
Exercise: Check!

Exercise Let $V = C[0, 1]$, with SCP $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ and induced norm. Compute $\|f\|$ for f with

$$f(t) = 1 - 3t^2, \quad f(t) = 5t - 3, \quad f(t) = t - t^3, \quad f(t) = t^3 - t^2.$$

Rules in a VS with induced norm With $\|v\| = \sqrt{\langle v, v\rangle}$ one derives several useful relations

1. **Pythagorean Theorem** $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ i.e. if $x \perp y$ we get $\|x+y\|^2 = \|x\|^2 + \|y\|^2$



   This follows from the definition $\|x+y\|^2 = \langle x+y, x+y\rangle = \dots$

2. **The inequality by Cauchy-Schwarz** $|\langle x, y\rangle| \leq \|x\|\|y\|$

   This needs little bit more thinking.

3. With $|\langle x, y\rangle| \leq \|x\|\|y\|$ we conclude

   (a) a definition of the angle α between two vectors x, y. It is α sucht that $\cos(\alpha) = \frac{|\langle x, y\rangle|}{\|x\|\|y\|}$ as $\frac{|\langle x, y\rangle|}{\|x\|\|y\|} \leq 1$,

   (b) and the triangle inequality $\Delta - \neq$ for an induces norm.

$$\begin{aligned} \|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

Exercises Verify

1. If the distance from u to v equals the distance from u to $-v$, then u and v are orthogonal.
2. $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$
3. If x and y are orthogonal unit vectors, then $\|x-y\| = \sqrt{2}$.
4. $\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \langle x, y\rangle$

Abstract distance and convergence With a norm $\|\cdot\|$ we have $\|x-y\|$ that leads to a notion of distance between $x, y \in V$. If a distance becomes minimal, we can speak of convergence:

Definition. *A sequence of vectors $(v_n)_{n\in\mathbb{N}}$ in a normed V converges to $v \in V$, if* $\lim_{n\to\infty} \|v_n - v\| = 0$.

Fact (Convergence independent of norm). *Let V be a* **finite dimensional** *vector space, i.e.* $\boxed{\dim V = n < \infty}$. *For two norms $\|\cdot\|_a$ und $\|\cdot\|_b$ it holds:*

$$\lim_{n\to\infty} \|v_n - v\|_a = 0 \iff \lim_{n\to\infty} \|v_n - v\|_b = 0.$$

Cave *That is not true in infinite dimensional VS!*

Bases and Coordinates in an Euclidean VS If a VS is equipped with a SCP (and an induced norm) the notions of linear independency and coordinates of a vector become handier.

Fact (**Orthogonal implies linear independence**). *Let $e_1, e_2, \ldots, e_k$ be unit vectors, that are pairwise orthogonal. Then $e_1, e_2, \ldots, e_k$ are* **linear independent** *and form a basis in an k-dimensional subspace.*

To very this start with $\alpha_1, ..., \alpha_k \in \mathbb{R}$ s.t. $\alpha_1 e_1 + ... + \alpha_k e_k \overset{*}{=} 0$. We must show that $\alpha_1 = ... = \alpha_k = 0$. For each $1 \leq i \leq k$ take SCP on both sides with the vector e_i: $\langle e_i, \alpha_1 e_1 + ... + \alpha_k e_k \rangle \overset{*}{=} \langle e_i, 0 \rangle = 0$. Orthogonal and unity of the e_i with the bilinear property of the SCP guarantee

$$\alpha_1 \underbrace{\langle e_i, e_1 \rangle}_{=0} + ... + \alpha_i \langle e_i, e_i \rangle + ... + \alpha_k \underbrace{\langle e_i, e_k \rangle}_{=0} = 0 \implies \alpha_i \underbrace{\langle e_i, e_i \rangle}_{=1} = \alpha_i = 0.$$

Hence for all $1 \leq i \leq k \rightsquigarrow \alpha_1 = ... = \alpha_k = 0$.

This works too, if e_i are simply orthogonal, length 1 is not needed or necessary: As $e_i \neq 0 \rightsquigarrow \langle e_i, e_i \rangle > 0$, the equation $\alpha_i \langle e_i, e_i \rangle = 0$ delivers $\alpha_i = 0$.

Definition (**Orthonormal basis**). *Let V be Euclidean and $\mathcal{B} = \{e_1, \ldots, e_n\}$ a basis of* **unit vectors**, *that are* **pairwise orthogonal**. *Such a basis $\mathcal{B}$ is called* **orthonormalbasis** *(ONB) and satisfies* $\langle e_i, e_j \rangle = \begin{cases} 1 & \text{falls } i = j \\ 0 & \text{falls } i \neq j \end{cases}$.

Example By our observations above we know that the functions $c_0, c_1, s_1, \ldots, c_N, s_N$ with

$$c_0(x) = \frac{1}{\sqrt{2\pi}}, \; c_k(x) = \frac{1}{\sqrt{\pi}} \cos(kx), \; s_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx),$$

form an ONB in $T_N \subset C^0([a, b])$.

Fact. *There's an algorithm that transforms a given basis of s finite-dimensional VS to an ONB of VS, i.e. every finite-dimensional VS has an ONB.*

Fact (**Coordinates w.r.t. ONB**). *For an ONB $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V we can write a vector $v \in V$ uniquely as* $\boxed{v = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e_i} \rangle e_i}$, *i.e. the k-coordinate w.r.t. ONB $\mathcal{B}$ of v is the coefficient $\langle v, e_k \rangle$.*

Because Let $v = \sum_{i=1}^{n} \alpha_i e_i$ be the unique representation w.r.t. ONB $\mathcal{B}$. Now we take the SCP of the given v and a basis vector e_k and get

$$\langle v, e_k \rangle = \left\langle \sum_{i=1}^{n} \alpha_i e_i, e_k \right\rangle \overset{*}{=} \sum_{i=1}^{n} \alpha_i \langle e_i, e_k \rangle \overset{**}{=} \alpha_k,$$

using $*$: Bilinearity of SCP and $**$: $\langle e_i, e_k \rangle = 0$ except $= 1$ if $i = k$.

Note As $\langle v, e_k \rangle$ = the k-coordinate of v w.r.t. ONB $\mathcal{B}$, it is much easier to computer coordinate vector $\varphi_{\mathcal{B}}(v)$. No solving of linear system is needed.

Exercises

1. What are the coordinates, if $\mathcal{B} = \{e_1, \dots, e_n\}$ are just orthogonal? Therefore, it is not necessary that the vectors have length 1.

2. Show in (a) and (b) that $\{u_1, u_2\}$ and in (c) and (d) that $\{u_1, u_2, u_3\}$ are orthogonal for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express x as a linear combination of the u's.

   (a) $u_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$

   (b) $u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

   (c) $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}$

   (d) $u_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

Application ONB to Fourier Let T_N be the subspace with ONB $c_0(x) = \frac{1}{\sqrt{2\pi}}, c_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx), s_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx)$. If $\boxed{f \in T_N}$ we get the coordinates using the SCP $f = \langle f, c_0\rangle c_0 + \sum_{k=1}^{N} \langle f, c_k\rangle c_k + \langle f, s_k\rangle s_k$

$$= \underbrace{\langle f, c_0\rangle \frac{1}{\sqrt{2\pi}}}_{=\frac{a_0}{2}} + \sum_{k=1}^{N} \underbrace{\langle f, c_k\rangle \frac{1}{\sqrt{\pi}}}_{=a_k} \cos(kx) + \underbrace{\langle f, s_k\rangle \frac{1}{\sqrt{\pi}}}_{=b_k} \sin(kx)$$

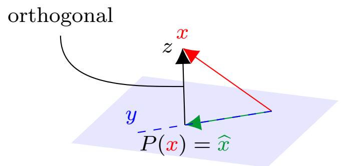
This construction returns our known formulae for the Fourier coefficients:

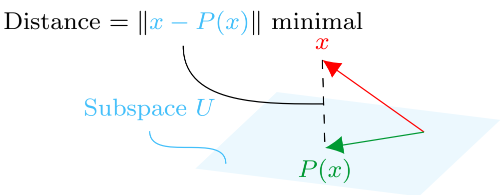
$$a_0 = \frac{2}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2\pi}} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \cos(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx) dx$$

$$b_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \sin(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx) dx$$

Orthogonal Projection as best approximation What can we do in the case of $f \notin T_N$? We use a construction that allows to project f onto the subspace T_N. Then we apply that this projection is the best possible approximation.





Fact (Orthogonal Projection onto vector or subspace). *Let V be a normed VS.*

1. *The projection of x onto a vector $y \neq 0$ is $P(x) = \frac{\langle x, y\rangle}{\langle y, y\rangle} y$.*
2. *Let U be a subspace of V and $e_1, \ldots, e_n$ an ONB of U.*

   Then $P(x) = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$ is the projection of $x \in V$ onto U. Among the vectors in U has the projection $P(x)$ the minimal distance from x.

Exercises

1. Compute the orthogonal projection of
   (a) $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto the line through $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and the origin.
   (b) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ onto the line through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and the origin.
2. Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of y onto $\langle\{u_1, u_2\}\rangle$.
   (a) For $y = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
   (b) For $y = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$
3. Let L be the subspace generated by y. Show that the projection is independent of basis vector.
4. Show: The projection is linear, i.e. $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y)$.

Application to Fourier series We are using this to get a new approach for the Fourier coefficients: Let $V = C^0([-\pi, \pi])$ with SCP $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ and $U = T_N$ the subspace with ONB $c_0, c_1, \ldots c_N, s_1, \ldots s_N$.
The projection $P_N(f)$ of $f \in V$ onto T_N is given by the construction:

$$P_N(f) = \langle f, c_0\rangle c_0 + \sum_{k=1}^{N} \langle f, c_k\rangle c_k + \langle f, s_k\rangle s_k \quad = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx).$$

The coefficients a_k and b_k are as above

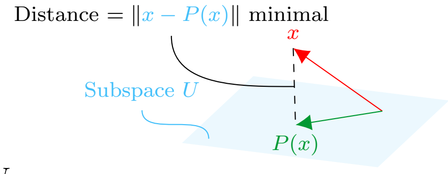
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

With increasing N, the approximation is improving, formally:

$$\|f - P_N(f)\|_{L^2}^2 = \int_{-\pi}^{\pi} (f - P_N(f))^2 dx \to 0 \text{ for } N \to \infty$$

Remarks on projection

We claimed above: $P(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ is the projection onto U and has minimal distance from x.

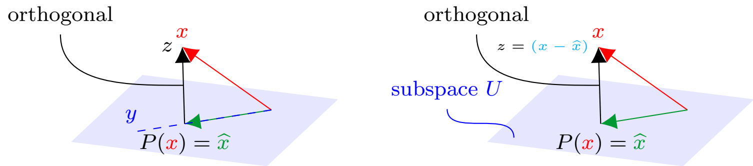


We want to see $x - P(x) \perp u$ for all $u \in U$.

$$\begin{aligned}\langle x - P(x), e_j \rangle = \langle x, e_j \rangle - \langle P(x), e_j \rangle &= \langle x, e_j \rangle - \langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, e_j \rangle \\ &= \langle x, e_j \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, e_j \rangle = \langle x, e_j \rangle - \langle x, e_j \rangle = 0\end{aligned}$$

How to conclude that distance between x and the projection $P(x)$ is minimal, i.e. $\|x - P(x)\| \leq \|x - u\|$ for all $u \in U$?
Let $u \in U \rightsquigarrow P(x) - u \in U \rightsquigarrow (x - P(x)) \perp (P(x) - u)$. We get a right-angled triangle with hypotenuse $x - u = (x - P(x)) + (P(x) - u)$, where we now apply Pythagorean's theorem.

Orthogonal projection and decomposition



- The orthogonal projection $P(x) = \widehat{x} = \dfrac{\langle x, y \rangle}{\langle y, y \rangle} y$ of $x \in V$ onto a vector $y \neq 0$ gives a decomposition $\boxed{x = \widehat{x} + z}$ with $z \perp y$.
- The projection $P(x) = \widehat{x} = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ on a subspace U gives also a decomposition $x = \widehat{x} + z$ with $z \perp U$. If $x \in U$ then $\widehat{x} = x$.

Orthogonal matrices An $m \times n$-matrix $A = (a_1 \quad a_2 \quad \cdots \quad a_n)$ is called **orthogonal** if it satisfies $A^T A = \mathbb{E}_n$. It has the following properties:

1. The column vectors a_i are pairwise orthogonal and $\|a_i\| = 1$.
2. For $x, y \in \mathbb{R}^n$ it yields $\|Ax\| = \|x\|$, $(Ax) \cdot (Ay) = x \cdot y$ and
$$(Ax) \cdot (Ay) = 0 \iff x \cdot y = 0,$$
i.e. the linear map $x \mapsto Ax$ preserves length and orthogonality.
3. Assume that the k column vectors of an orthogonal $n \times k$-matrix A form a basis of a subspace U of $\mathbb{R}^n$.
Then the orthogonal projection of a vector $x \in \mathbb{R}^n$ is $\widehat{x} = AA^T x$.

Fact (**(Reduced) QR-decomposition).** *Let A be an $m \times n$-matrix with* **linearly independent columns.**

There exists an orthogonal $m\times$)-matrix Q s.t. $\boxed{A = QR}$ *where R is an upper triangular and invertible $n \times n$-matrix with positive entries on its diagonal.*

Part III

Appendix

Eigenvalues and -vectors

In the course of our investigations the concept of eigenvalues and eigenvectors has an crucial impact and application. Therefore it is appropriate to recall and collect essential facts in more detail. First recall

Definition (Eigenvalues/-vectors). *Let A be a quadratic matrix.*

1. *A number λ is called eigenvalue of A* **(EVal)** *if there is a vector $v \neq 0$ such that* $\boxed{A \cdot v = \lambda v}$.
2. *Each vector $v \neq 0$ is an eigenvektor of A* **(EVec)** *with respect to EVal λ, if the equation* $\boxed{A \cdot v = \lambda v}$ *is fulfilled.*

Now we gather known properties and results. We don't distinguish whether it is a theorem or a simply conclusion.

Facts (Eigenvalues/-vectors). *Let $A = (a_{ij})$ be a quadratic matrix.*

Eigenspaces

1. *If v_1, v_2 are EVec with the* **the same** *EVal λ the vector $w = \alpha v_1 + \beta v_2$ is also an EVec with EVal λ. With $Eig(\lambda)$ we denote the subspace generated by eigenvectors with respect to eigenvalues λ.*
2. *EVec with respect to* **different** *EVal are linear independent.*
3. *A basis formed by eigenvectors is called* **eigenbasis***.*

Finding EVal and EVec

4. *We find EVec with EVal λ as solutions $x \neq 0$ of a homogeneous system of linear equations $(A - \lambda \cdot E_n) \cdot x = 0$.*
5. *The EVal of a matrix A are exactly the zeroes of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda \cdot E_n)$.*
6. *As $p_A(\lambda)$ has degree n the fundamental theorem of Algebra implies that A has n (complex) EVal λ_i with a multiplicity. Some of the λ_i might be real. Thus $p_A(\lambda) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$.*

Handy relations

7. *The sum of the diagonal entries* a_{ii} *equals the sum of the EVal:*

$$\sum_i a_{ii} = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

8. *Moreover* $\boxed{\det(A) = \lambda_1 \cdot \lambda_2 \ldots \lambda_n}$ *and therefore: The inverse* A^{-1} *exists, if and only if all* $\lambda_i \neq 0$.

9. *If we have an EVec* v *with* λ *and in addition* A *is* **invertible**, *then* v *is an EVec of* A^{-1} *with EVal* $\frac{1}{\lambda}$.

10. *Assume that* **all** *coefficients of* $p_A(\lambda)$ *are* **real**, *e.g. in the case of* $a_{ij} \in \mathbb{R}$ *only real entries. If* λ_k *is an EVal then the complex conjugate* $\overline{\lambda_k}$ *is an EVal, too.*

Application to a discrete model $v_{n+1} = Av_n$ Assume that we have a model situation where we get a collection of m values or magnitudes $x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}$ after n timesteps. This assignment is a map $n \mapsto v_n = \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_m^{(n)} \end{pmatrix}$. If we wait one time unit (1 TU) we get the vector $v_{n+1} = \begin{pmatrix} x_1^{(n+1)} \\ \vdots \\ x_m^{(n+1)} \end{pmatrix}$.

What is the relation $v_n \rightsquigarrow v_{n+1}$? Let us assume this transition is linear, i.e. there is a $m \times m$-matrix A, such that $v_{n+1} = Av_n$. We have the following diagram that describes the situation.

$$\begin{array}{ccc} n & \xrightarrow{\textbf{explicit}} & v_n \in \mathbb{R}^m \\ {\scriptstyle 1\text{ TU}}\downarrow & & \downarrow{\scriptstyle \textbf{recursive}} \\ n+1 & \longrightarrow & v_{n+1} = Av_n = A^{n+1}v_0 \in \mathbb{R}^m \end{array}$$

The **recursive** representation means that for determination of the vector v_{n+1} one needs all the vectors $v_n, v_{n-1}, \ldots, v_0$ and must compute $n+1$ times a matrix-vector-product $v_i = Av_{i-1}$. In the **explicit** case the vector v_n just depends on n. Here one needs powers of the matrix A If we continue the iteration backwards, we get:

$$\begin{aligned} v_{n+1} &= Av_n = A(Av_{n-1}) = A(A(Av_{n-2})) = \ldots = \overbrace{A(A(A(\ldots A}^{n+1 \text{ times}}\, v_0)))\ldots) \\ &= A^{n+1}v_0 \end{aligned}$$

Using the calculation rules of matrix multiplication, it becomes

$$\underbrace{A(A(\ldots A)\ldots)}_{n+1 \text{ times}} = \underbrace{(AAA\ldots A)}_{n+1 \text{ times}} = A^{n+1}.$$

It results in an explicit representation by $v_n = A^n v_0$.

What can we say about the sequence (v_n)?

We might ask ourselves how the values $x_i^{(n)}$ develop over time, for example

1. Is there an equilibrium, a monotony or an asymptotic behaviour?
2. Are there starting values $x_i^{(0)}$ at which nothing happens? That is,
$$x_i^{(0)} = x_i^{(1)} = x_i^{(2)} = \dots \quad i = 1, 2, \dots m$$
3. Is it possible that the values vary, but the ratio does not?
$$\frac{x_i^{(n)}}{x_j^{(n)}} = \frac{x_i^{(n+1)}}{x_j^{(n+1)}},\ x_j^{(n+1)} \neq 0 \neq x_j^{(n)}?$$

Example/Exercise Check that for $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 6 & 0 & -6 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ we have $A^3 = E_3$.

For a sequence $v_n = Av_{n-1} = A^n v_0$ we get a periodic behaviour:

$$v_3 = A^3 v_0 = v_0 \text{ and thus } v_{n+3} = v_n.$$

What does $A^3 = E_3$ mean for the eigenvalues of the matrix A?

Note that here $m = k = 3$. In general, $k \neq m$ might also be the case.

Further examples Here is a list of numerical examples done by the computer. We want to understand and investigate those results by using EVal and EVal.

	Matrix	Initial vector	Series
1.	$\begin{pmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$	Cycle with 3 TU
2.	$\begin{pmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$	$\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$	equilibrium $\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$
3.	$\begin{pmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$	convergence to equilibrium $\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$
4.	$\begin{pmatrix} 0 & 2 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	Each sequences (x_{n+1}/x_n),(y_{n+1}/y_n) and (z_{n+1}/z_n) converges to a fixed number λ. The sequence $(v_n/\lvert v_n\rvert)$ of the normalised v_n converges.

We leave the analysis of first two examples as an exercise, see below. In the 3. example we have complex EVal, that will be studied after the 4. example.

Example with dominant EVal With $A = \begin{pmatrix} 0 & 2 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$ and $v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ we define an iteration: $w_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = Aw_n = A^{n+1} w_0$.

Why do the sequences of the quotients (x_{n+1}/x_n), (y_{n+1}/y_n), (z_{n+1}/z_n) each converge to a fixed number λ?

For large n we get $x_{n+1}/x_n \approx \lambda,\ y_{n+1}/y_n \approx \lambda,\ z_{n+1}/z_n \approx \lambda$ or

$$x_{n+1} \approx \lambda \cdot x_n,\ y_{n+1} \approx \lambda \cdot y_n,\ z_{n+1} \approx \lambda \cdot z_n$$

and as vectors $Aw_n = w_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} \approx \begin{pmatrix} \lambda x_n \\ \lambda y_n \\ \lambda z_n \end{pmatrix} = \lambda \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \lambda w_n$. This means that for increasing n the vector w_n becomes more and more an EVec of A of the EVal λ, and that the vectors $w_0, w_1, w_2, \ldots$ point more and more in the direction of an EVec.

To confirm this we use the characteristic polynomial $p_A(\lambda) = -\lambda^3 + \lambda + \frac{3}{8}$ (Check this.). It provides the eigenvalues $\lambda_1 = -\frac{1}{2}$ and $\lambda_{2/3} = \frac{1}{4}(1 \mp \sqrt{13})$ (Check again). As eigenvectors we choose

$$v_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 7-\sqrt{13} \\ 1-\sqrt{13} \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 7+\sqrt{13} \\ 1+\sqrt{13} \\ 1 \end{pmatrix}.$$

Check, that indeed $Av_i = \lambda_i v_i$ for $i = 1, 2, 3$.

Application to $v_{n+1} = Av_n$

1. The three EVal are distinguished, i.e. $v_1, v_2, v_3 \in \mathbb{R}^3$ form an eigenbasis.
2. Each initial vector w_0 can be written as $w_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where α_1, α_2 and α_3 are unique
3. After n TU with the linear matrix-vector-product we get
$$w_n = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2 + \alpha_3 \lambda_3^n v_3.$$
4. Since $|\lambda_1|, |\lambda_2| < 1$ and $\lambda_3 > 1$, then $\lambda_1^n, \lambda_2^n \to 0$ and $\lambda_3^n \to \infty$.
5. If $\alpha_3 = 0$, then $w_n \to 0$, e.g. $w_0 = v_1 + v_2 = \begin{pmatrix} 9-\sqrt{13} \\ -1-\sqrt{13} \\ 2 \end{pmatrix}$. If $\alpha_3 \neq 0$, it follows that the sequence $w_0/|w_0|,\ w_1/|w_1|,\ w_2/|w_2|, \ldots$ of normalised vectors converges to the normalised EVec $v_3/|v_3|$ corresponding to the EVal λ_3.
6. This applies to any choice of EVec $v_1, v_2, v_3 \in \mathbb{R}^3$.

Example complex EVal Let $A = \begin{pmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$ and $v_0 = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$. The computer shows a convergence to equilibrium $\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$ of the sequence (v_n) with $v_{n+1} = Av_n$. We validate this by using EVal and EVec.

The characteristic polynomial $p_A(\lambda) = -\lambda^3 + \dfrac{1}{2}\lambda + \dfrac{1}{2} = (\lambda - 1)\Big(-\lambda^2 - \lambda - \dfrac{1}{2}\Big)$ gives the EVal $\lambda_1 = 1$ and $\lambda_{2,3} = -\frac{1}{2} \pm \sqrt{-\frac{1}{4}}$ or

$$\lambda_1 = 1 \in \mathbb{R}, \quad \lambda_2 = -\frac{1}{2} + \frac{1}{2}i \in \mathbb{C} \quad \text{and} \quad \lambda_3 = -\frac{1}{2} - \frac{1}{2}i \in \mathbb{C},$$

and the absolut value of the EVal $\lambda_{2/3}$ is

$$|\lambda_2| = \left|-\frac{1}{2} + \frac{1}{2}i\right| = \sqrt{\frac{1}{2}} = |\lambda_3| = \left|-\frac{1}{2} - \frac{1}{2}i\right| < 1.$$

What does this mean for the iteration? If we write $\lambda_2 = re^{i\varphi}$ in polar coordinates with $r = |\lambda_2| < 1$, then $r^n \to 0$ for $n \to \infty$. The absolute value becomes smaller and smaller and the sequence of complex numbers $\lambda_2, \lambda_2^2, \lambda_2^3, \ldots$ spirals into the origin. Analogue considerations work for λ_3. Let's also choose eigenvectors

$$v_1 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -3i \\ -\frac{3}{2} + \frac{3}{2}i \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 3i \\ -\frac{3}{2} - \frac{3}{2}i \\ 1 \end{pmatrix}.$$

If we apply and use this, we see

1. The three EVal are pairwise different from each other and three EVec give us an eigenbasis. That allows to write each initial vector w_0 in the form $w_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. Therefore $w_n = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2 + \alpha_3 \lambda_3^n v_3$.

2. We have $\lambda_1 = 1$ and $|\lambda_2| < 1, |\lambda_3| < 1$. This means that $\lambda_1^n = 1$ is constant for all n and for the other two sequences $\lambda_2^n, \lambda_3^n \to 0$.

3. If $\alpha_1 = 0$, then $w_n \to 0$.
   If $\alpha_1 \neq 0$, then $w_n \to \alpha_1 \lambda_1^n v_1 = \alpha_1 v_1$, i.e. w_n converges to the vector αv_1. The computer shows for example for the start vector $w_0 = \left(\begin{smallmatrix} 0 \\ -3 \\ 2 \end{smallmatrix}\right) = v_2 + v_3$, that the sequence of vectors $w_1, w_2, w_3, \ldots, w_n, \ldots$ converges to the zero vector.
   With initial $w_0 = \left(\begin{smallmatrix} 10 \\ 10 \\ 10 \end{smallmatrix}\right)$, the sequence $w_1, w_2, w_3, \ldots, w_n, \ldots$ converges to the vector $\left(\begin{smallmatrix} 24 \\ 12 \\ 4 \end{smallmatrix}\right) = 4\left(\begin{smallmatrix} 6 \\ 3 \\ 1 \end{smallmatrix}\right)$. In order to get the coefficient a priori, we must use the equation $w_0 = \left(\begin{smallmatrix} 10 \\ 10 \\ 10 \end{smallmatrix}\right) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, and determine the coefficient α_1, i.e. solving $C\alpha = b$ with

   $$C = \begin{pmatrix} 6 & -3i & 3i \\ 3 & -\frac{3}{2} + \frac{3}{2}i & -\frac{3}{2} - \frac{3}{2}i \\ 1 & 1 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}.$$

   One findes $\alpha = \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 \\ 3 - \frac{7}{3}i \\ 3 + \frac{7}{3}i \end{smallmatrix}\right)$.

Summary Given an $m \times m$-matrix A, that defines $w_{n+1} = Aw_n$.

Let us assume that there are m linearly independent EVec of A: $v_1, v_2, \ldots, v_m$, with EVal $\lambda_1, \lambda_2, \ldots, \lambda_m$. The λ_i are not necessarily different from each other.

Let w_0 be a initial vector. Since the EVec v_i are linearly independent, there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ with $w_0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$. After n time units: $w_n = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2 + \ldots + \alpha_m \lambda_m^n v_m$.

1. If $|\lambda_i| < 1$ holds for each EVal, then $\lambda_i^n \to 0$ and therefore $w_n \to 0$.

2. For example, let $\lambda_1 = 1$, and for all other EVal λ_i let $|\lambda_i| < 1$ apply.
   Then $\lambda_1^n = 1$ and $\lambda_i^n \to 0$ and therefore $w_n \to \alpha_1 v_1$.
3. For example, let $\lambda_1 > 1$ be a real EVal and $|\lambda_i| < 1$ for all other EVal λ_i. Then follows $\lambda_1^n \to \infty$ and for the other EVal $\lambda_i^n \to 0$.
4. More generally again: For example, let λ_1 be an EVal with $\lambda_1 > |\lambda_i|,\ i = 2, 3, 4, \ldots, m$. For each starting $w_0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$ with $\alpha_1 \neq 0$ the vectors $w_1, w_2, \ldots, w_n, \ldots$ approach the direction of v_1 more and more. This means again: The sequence $(w_n/|w_n|)_n$ of the normalised vectors converges towards the normalised EVec $v_1/|v_1|$.

Exercises

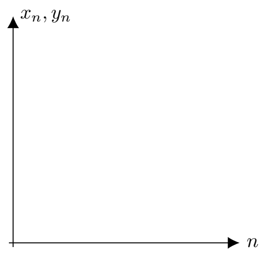
1. Verify the examples

	Matrix	Initial vector	Series
1.	$\begin{pmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$	Cycle with 3 TU
2.	$\begin{pmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$	$\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$	equilibrium $\begin{pmatrix} 24 \\ 12 \\ 4 \end{pmatrix}$

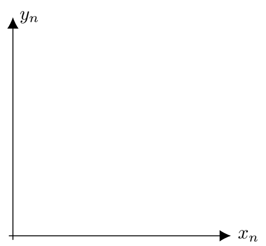
2. Let $v_{n+1} = \binom{x_{n+1}}{y_{n+1}}$, $v_n = \binom{x_n}{y_n}$ and $A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{13}{10} \end{pmatrix}$.

   (a) Compute (with a CAS or by hand) the vectors v_n, for $n = 1, 2, 3 \ldots$ with initial vectors $v_0 = \binom{1}{2}$ and $v_0 = \binom{4}{3}$. What can you say about these vectors?

   (b) Start now with $v_0 = \binom{400}{5000}$ and $v_0 = \binom{5000}{5000}$ and plot for $n = 1, 2, 3 \ldots$ the values of x_n and y_n in the same coordinate system.



   (c) Plot for both initial vectors in (b) the vector v_n into a (x_n, y_n)-coordinate system.



   (d) Write $\binom{400}{5000}$ and $\binom{5000}{5000}$ (using a CAS) as a linear combination of $\binom{1}{2}$ and $\binom{4}{3}$ and explain your observations in (c).

1st order ODE

We start with examples for (ordinary) differential equations.

Let $t \mapsto N(t)$ be a continuous model with the initial value $N(0) = N_0$ and a constant rate, i.e. for all t we have $N'(t)/N(t) = r$. If we rewrite the equation, we get $N`(t)/N(t) = r \implies N'(t) = r \cdot N(t)$ or in short $N' = rN$. The is a first example of an ordinary differential equation (ODE). The equation is characterised by the following:

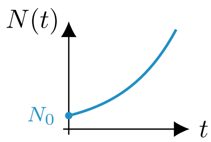
A solution is a function f with the property that the derivative f' is equal to the initial function f multiplied by a constant factor r. We are therefore looking for a function and not a number that fulfils this equation as the solution of an ODE. In contrast to an equation of the form $N'(t) = \frac{1000}{1+t/2}$ the equation $N' = r \cdot N$ also contains the function $N : t \mapsto N(t)$ on the right-hand side. So we cannot simply look for a primitive for the right-hand side:

$$N(t) = \int N'(t)dt = \int \frac{1000}{1+\frac{1}{2}t}dt = 2000 \ln\Big(1 + \frac{1}{2}t\Big) + C,\ C = \text{constant}.$$

For an ordinary differential equation (ODE) the solution is a function in a variable. For a partial differential equation (PDE), the solution is a function in several variables.

If we choose $r = 1$, we are looking for a function N with $N' = N$. We recall (from Calculus) that this applies to the exponential function, i.e. $(e^t)' = e^t$. Thus in detail, if $N(t) = e^t$ we get $N'(t) = (e^t)' = e^t = N(t)$.

Is this the only possibility? For a constant c we consider $N(t) = c \cdot e^t$. The function also fulfils the ODE, because $N'(t) = (c \cdot e^t)' = c \cdot (e^t)' = c \cdot e^t = N(t)$.



To find a unique solution, we use the initial value $N(0) = N_0$, defining an initial value problem (IVP) in which we have a ODE $N' = N$ and in addition an initial value $N(t_0) = N_0$.

For $t_0 = 0$ we get $N(t_0) = N(0) = c \cdot e^0 = c$, i.e. $N(t) = N_0 \cdot e^t$. In the same way, for $N' = r \cdot N$ the solution is $N(t) = N_0 \cdot e^{rt}$ with the known graph above.

Exercise Find $N(t)$ and sketch the graph in case of $t_0 > 0$.

Uniqueness When modelling it is useful or perhaps imperative that we work with an unique solution, or at least be able to decide whether this is the case. Statements about the existence and uniqueness of a solution require more theory.

As an elementary example, let us look at exponential ODE and ask ourselves why the exponential function provides the only solution to the equation $N' = N$?

Let f be another function that fulfils the equation $f' = f$. Let g be an auxiliary function with $g(t) = f(t) \cdot e^{-t}$. This function measures, how the values $f(t)$ differs from e^t. Now we compute

$$\begin{aligned} g'(t) &= \left(f(t) \cdot e^{-t}\right)' = f'(t) \cdot e^{-t} + f(t) \cdot (-e^{-t}) && \text{product and chain rule} \\ &= f'(t) \cdot e^{-t} - f'(t) \cdot e^{-t} && f' = f \\ &= 0. \end{aligned}$$

If the derivative vanishes everywhere the function g, i.e. $g(t) = C$ must be constant. Due to the definition of the auxiliary function $g(t) = f(t) \cdot e^{-t} = C$ we conclude that indeed $f(t) = C \cdot e^t$. Therefore that f is again an exponential function. With the initial $f(0) = N_0$ moreover $f(t) = N_0 \cdot e^t$.

The same applies for $N' = rN$ with initial value $N(t_0) = N_0$. Here too, the exponential function provides the unique solution $f(t) = N_0 \cdot e^{rt}$.

Bounded growth Exponential growth is an idealisation. In application there is a limitation and therefore we must introduce a correction. We obtain an ODE of the form $N'(t) = r(K - N(t))$, $0 < N(0) = N_0 < K$ and $r > 0$. This is another example of a *linear* ODE. A solution of this equation is again a function f with the property : On the left-hand side of the equation is the derivative f', and if we insert f on the right-hand side, we get $r(K - f)$, i.e. for each t from the common definition sets of D_f and $D_{f'}$ one has $f'(t) = r(K - f(t))$. We are again looking for a function and not a number that fulfils this differential equation.

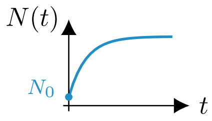
Why does this equation describe a development with limited growth? We take a qualitatively look at the solution behaviour without knowing the solution explicitly. We assume here and in the following that a solution exists and is unique.

1. As $0 < N(0) < K$ it implies $K - N(0) > 0$. With $r > 0$ we see a positive slope at the initial value $N'(0) = r(K - N(0)) > 0$ and further $N'(t) = r(K - N(t)) > 0$, as long as $N(t) < K$. The function N is strictly monotonically increasing as long as $N(t) < K$.
2. Assume that $N(t)$ is much smaller than K at the beginning, i.e. for small t it holds $K - N(t) \approx K$. The ODE tells us that $N'(t) = r(K - N(t)) \approx rK$, i.e. constant growth and can solve the equation $N' \approx rK$ to see a linear model with $N(t) \approx \displaystyle\int rK dt = (rK) \cdot t + C$.
3. With $N(t) \to K$ follows $K - N(t) \to 0$, and we get slower and slower growth, since the right-hand side of the ODE $N'(t) = r(K - N(t))$ becomes smaller and smaller, the velocity N' becomes smaller and smaller. For the value $N_\infty = K$ the ODE is $N'_\infty = r(K - K) = 0$, and there is no more

growth. Thus K is an upper bound (maximum capacity). The values of N grow over time and approache the asymptote $N_\infty = K$ more and more.

4. How does the growth rate N' change? Where are any inflection points? To do this, we calculate $N''(t) = (N')'(t) \overset{\text{ODE}}{=} (r(K - N(t)))' = -rN'(t) \neq 0$ as $r > 0$ and $N'(t) > 0$, there are no inflection points.

The solution is $N(t) = (N_0 - K) \cdot e^{-rt} + K$ with the graph.



We check whether this function is indeed a solution:

$$N(0) = (N_0 - K) \cdot e^{-r0} + K = (N_0 - K) \cdot 1 + K = N_0 - K + K = N_0$$

and

$$\begin{aligned}
N`(t) &= \big((N_0 - K) \cdot e^{-rt} + K\big)' = (N_0 - K) \cdot \big(e^{-rt}\big)' && \text{Derivative} \\
&= (N_0 - K) \cdot (-r)e^{-rt} && \text{Chain rule} \\
&= (N_0 - K) \cdot (-r)e^{-rt} - rK + rK && \text{Expand with zero} \\
&= r\big(-((N_0 - K) \cdot e^{-rt} + K)\big) + rK && \text{Arithmetics} \\
&= r\big(-N(t)\big) + rK = r\big(K - N(t)\big) && \text{Definition } N(t).
\end{aligned}$$

Linear ODE with constant coefficients Exponential and bounded growth are examples of linear ODEs with constant coefficients. In general we say

Definition. *An nonhomogeneous linear ODE with* constant coefficients *is an equation of the form $y' = ay + b$ with constants a and b.*

If $b = 0$, the ODE is homogeneous.

Our first examples are of this kind, i.e. for $N' = rN$ we have $N = y, a = r$ and $b = 0$. And for the bounded growth $N' = r(K - N)$, we compute $N' = rK - rN = -rN + rk$, i.e. $a = -r$ and $b = rK$.

Solutions of a linear ODE with constant coefficients We now specify different cases in which functions solve a such an ODE.

Primitive

If $a = 0$ we have $y' = b$, which is solved directly: $y = \displaystyle\int b\, dt = b \cdot t + C$.

Stationary Solution

A stationary solution of $y' = ay + b$ is a constant solution y_∞ with $y'_\infty = 0$. This for $a \neq 0$ we see $0 = ay_\infty + b \implies y_\infty = -\dfrac{b}{a}$.

Homogeneous Solution

If $b = 0$, the ODE is homogeneous, we are in the exponential case with the general solution $y_H(x) = C \cdot e^{ax}$.

General solution

A stationary solution combined with the *homogeneous* solution provides the general solution of $y' = ay + b$ by $y(x) = y_H + y_\infty = C \cdot e^{ax} - \frac{b}{a}$. Let us check this

$$\begin{aligned} y'(x) &= \Big(C \cdot e^{ax} - \frac{b}{a}\Big)' = Ca \cdot e^{ax} && \text{Chain rule} \\ &= a\Big(C \cdot e^{ax}\Big) = a\Big(C \cdot e^{ax} - \frac{b}{a} + \frac{b}{a}\Big) && \text{Extension by zero} \\ &= a\Big(y(x) + \frac{b}{a}\Big) = ay(x) + b && \text{Insert } y(x) = C \cdot e^{ax} - \frac{b}{a} \end{aligned}$$

Initial Value Problem

If we specify an initial value $y(0) = y_0$, we obtain for the constant C above

$$y(0) = y_0 = C \cdot e^0 - \frac{b}{a} = C - \frac{b}{a} \implies C = y_0 + \frac{b}{a}$$

and after inserting $y(x) = \Big(y_0 + \frac{b}{a}\Big) \cdot e^{ax} - \frac{b}{a} = y_0 \cdot e^{ax} + \frac{b}{a}\left(e^{ax} - 1\right)$. If we apply this formula to the limited growth, we get $a = -r$ and $b = rK$. This values results $\pm\frac{b}{a} = \mp K$. Using this and $y_0 = N_0$ in the above formula show that we get indeed $N(t) = (N_0 - K) \cdot e^{-rt} + K$.

Convergence

The general solution $y(x) = C \cdot e^{ax} - \frac{b}{a}$ also makes a direct statement about the convergence behaviour. For $a < 0$ there is convergence $C \cdot e^{ax} \to 0$ for $x \to \infty$ and thus $y(x) \to -\frac{b}{a} = y_\infty$. Because $a < 0$, y_∞ is then positive if b is also positive. In the case $a > 0$ we have no convergence for $x \to \infty$.

In the above example of limited growth we have $-\frac{b}{a} = K$ as an asymptote.

Exercises

1. Let f with $f(0) = 1$ be the solution of $y' = ay + 2023$.
   Which a gives $\lim\limits_{x \to +\infty} f(x) = 119$?

2. Solve and plot the solution for $y'(x) \pm 2y(x) = 1$.

Overview for ODE in general An ordinary differential equation (ODE) is an equation of the form $y^{(n)} = F\big(x, y, y', ..., y^{(n-1)}\big)$. It consists of

- an unknown function in *one* variable, which we can denoted by $y = y(x)$ or $f = f(x)$ or also $x = x(t)$ ...

- derivatives y‘, y'', ..., $y^{(5)}$, ... $y^{(n)}$, ... or f' or also $\dot{x}$, $\ddot{x}$ ...
- and the variable x or t ...

The order of an ODE $y^{(n)} = F(x, y, y', ..., y^{(n-1)})$, denotes the highest occurring derivative. We are particularly interested in the 1st order, as these are important for growing processes. These can be linear like $y'(x) = -ay(x)$ or non-linear like the logistic ODE of the form $y'(x) = ay(x)(B - y(x))$.

Differential equations of the 2nd order mainly occur in oscillation processes and have the form for example, have the form $my''(x) = -ky(x) - ry'(x)$. This is the equation of motion of a harmonic oscillation *with friction*.

Solutions of an ODE We are looking for a function $f : D \to \mathbb{R}, x \mapsto f(x)$ defined on an interval D that has the property: If, for each $x \in D$ on the left-hand side of the equation $y^{(n)} = F(x, y, y', ..., y^{(n-1)})$ the corresponding derivative $f^{(n)}(x)$, this must be equal to of the right-hand side if we insert the corresponding derivatives there. The equation must be fulfilled for *every* $x \in D$, not just for individual x. A solution can be given in different ways. We distinguish

General solution and Initial Value problem The *general solution* of an ODE depends on constants. If the order is n, these are n constants, i.e. in case of a 1st-order the general solution contains one constant C_1, and for 2nd order there are two constants C_1, C_2. These constants can be determined by the initial values $f(x_0), f'(x_0), ..., f^{(n-1)}(x_0), ...$

Note that if the order is n, we need n initial values, even if not all derivatives $y^{(n)}$ occur in the ODE $y^{(n)} = F(x, y, y', ..., y^{(n-1)})$. So for 2nd order, both constants C_1 and C_2 must be defined, even if no 1st derivative appears, e.g. $my''(x) = -ky(x)$, the equation for the harmonic oscillation *without friction*, in which the 1st derivative is missing.

Stationary solutions The solutions y_∞ of the ODE that are independent of the variable x are called stationary solutions or equilibrium solutions or also fixed point. If the right-hand side of $y^{(n)} = F(y, y', ..., y^{(n-1)})$, i.e. independent of the variable x, then y_∞ is constant.

Stationary solutions of a 1st order ODE $y' = F(y)$ Here it is important that the right-hand side $F(y)$ only depends on y. The following applies: y_∞ is a stationary solution $\iff$ y_∞ is constant $\iff$ $y'_\infty = F(y_\infty)$ is constant zero. Equilibrium solutions are therefore just the zeros of F and in a stationary solution, the population is in equilibrium, i.e. t $y'_\infty = 0$, so there is no change.

Due to the uniqueness of the solution of the IVP, two solutions do not intersect. Therefore, stationary solutions are candidates for asymptotes. For a stationary solution the system is in an equilibrium $y'_\infty = 0$,

Exercise Let $y'(x) = (y(x) - 1)(y(x) + 1)(y(x) - 2)$. For which $y(0) = y_0$ is the solution constant? For which strictly monotonous increasing/decreasing?

We investigate the question of how solutions behave in the vicinity of an equilibrium solution: Do they converge towards y_∞, or are they repelled by y_∞? Can we specify a criterion for convergence near y_∞?

Statements about the existence and uniqueness of a solution require more theory. We assume here that we have a unique AWP solution in each case. Then we know that two solutions with AWP $y_0 \neq y_1$ do not intersect, and there is at most there is an asymptotic approximation.

Often only a numerical solution or a qualitative description of the solution is possible. Depending on the form, it is possible to specify a formula in elementary functions. The main methods are then the separation of the variables or integrating factors.

Linear ODE in general Let $p, q : \mathbb{R} \to \mathbb{R}$ be two functions. An ODE of the form $y'(x) = p(x)y(x) + q(x)$ is called **linear** ODE (of 1st order). If $q = 0$ it is called **homogenous**.

Exercise Which ODE are linear?

1. $y'(x) = x - y(x) + 1$
2. $y'(x) = \dfrac{y(x)}{x} + \sin(x)$
3. $y'(x) = \dfrac{x}{y(x)}$
4. $y'(x) = \dfrac{y(x)}{x} + y(x)$
5. $y'(x) = \dfrac{y(x)}{x} + \sin(y(x))$

Solution with particular solution Let y_H be the general solution of the homogenous ODE $y'(x) = p(x)y(x) \rightsquigarrow y_H = K \cdot e^{P(x)}$. Let y_{sp} be a particular solution of $y'(x) = p(x)y(x) + q(x)$. Then $y = y_H + y_{sp}$ general solution of **nonhomogenous ODE** $y'(x) = p(x)y(x) + q(x)$. **Exercise** Check!

Note that is a generalisation of the above constant case $y' = ay + b$ with solution given by $y(x) = \underbrace{C \cdot e^{ax}}_{y_H} + \underbrace{\left(-\frac{b}{a}\right)}_{y_{sp} = y_\infty}$.

To get solution y_{sp} one could apply different Ansätze (e.g. slope field below). Mostly Integrating factors is more target-oriented.

Integrating Factors Let $y'(x) = p(x)y(x) + q(x)$.

Method of Integrating Factors delivers general solution:

$$y(x) = (K_0(x) + C)e^{P(x)} = K_0(x)e^{P(x)} + Ce^{P(x)}$$

where $K_0(x) \in \displaystyle\int q(x)e^{-P(x)}$ and $P' = p$.

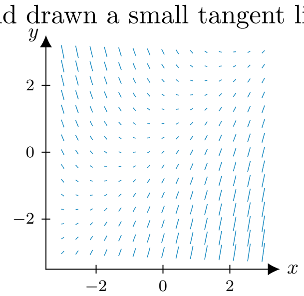
Note that in the sum $K_0(x)e^{P(x)} + Ce^{P(x)}$ we have $K_0(x)e^{P(x)}$ as a particular solution y_{sp} of **nonhomogenous** $y'(x) = p(x)y(x) + q(x)$, i.e. we choose $C = 0$. And $Ce^{P(x)}$ is the general solution y_H of the homogeneous part. Thus the general solution of **nonhomogenous** $y'(x) = p(x)y(x) + q(x)$ can always be split into $y = y_H + y_{sp}$.

Exercises Solve and plot the solution.

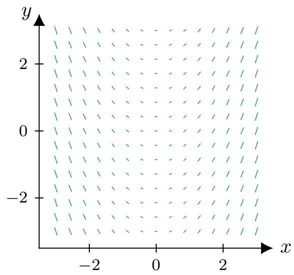
1. $y'(x) = y(x) + x$.
2. $y'(x) = y(x) + \sin(x)$ with $y(0) = 1$.
3. $y'(x) - y(x) = xe^{-x}$.

Slope Field Given $y'(x) = F(x, y)$ and a solution f with $f(x) = y$.

For $(x, f(x)) = (x, y)$ on the graph the slope of the tangent at the point (x, y) is given by $f'(x)$, and with the differential equation this is $f'(x) = F(x, y)$. Thus if we look at are at the point (x, y) and this lies on a solution f of $y'(x) = F(x, y)$, we obtain the value of $f'(x)$ by inserting the coordinates x and y on the right-hand side $F(x, y)$. In the following two figures, the computer has calculated the gradient and drawn a small tangent line.



$y' = x - y + 1$



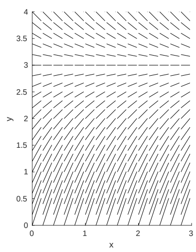
$y' = x$

The slope field gives us a qualitative idea of what a solution curve looks like.

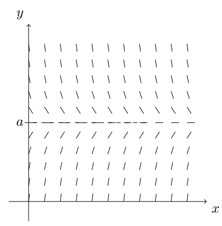
On the right, we recognise the parabolic contour and thus directly the solution by $y(x) = \displaystyle\int x\,dx$. On the left we see the bisector $y = x$. In fact, this fulfils the function $y(x) = x$ fulfils the differential equation, since $y'(x) = 1 = x - x + 1$.

Exercise

1. Determine the missing a and b.



$y'(x) = -2y(x) + b$. What is b?

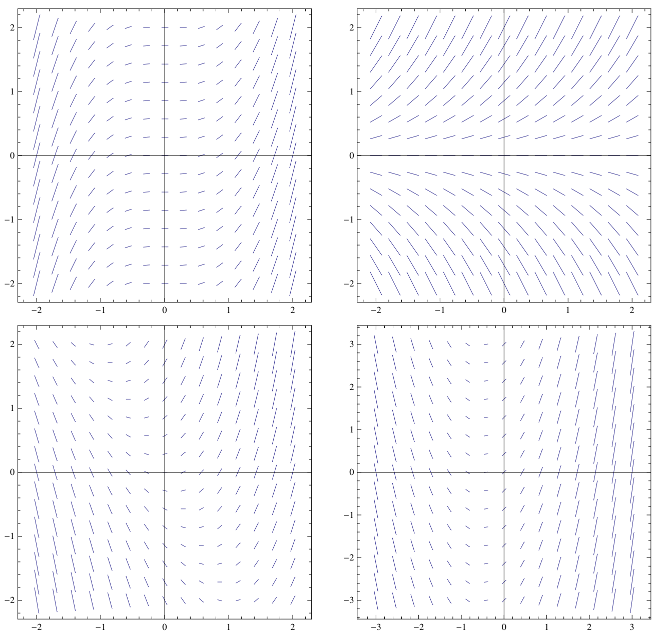


$y'(x) = -4y(x) + 8$. What is a?

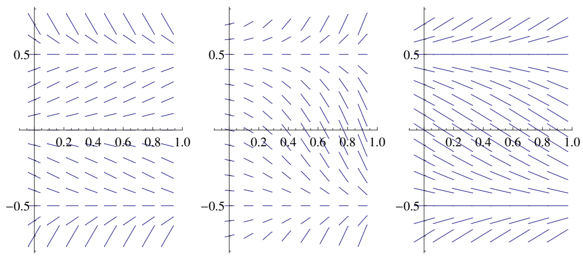
2. Match each of the ODEs

$$y'(x) = y(x), \quad y'(x) = 2x + y(x), \quad y'(x) = x^2, \quad y'(x) = 2x + 1$$

to a slope field.



3. Which slope field belongs to the ODE $y'(x) = y^2(x) - \dfrac{1}{4}$?



2nd order ODE

Using an example (with double EVal) we explain that solving a 2×2-ODE-system leads to solving a 2nd order differential equation. Let $y'(x) = A \cdot y(x)$ be given with $y'(x) = \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$. Then A has the double EVal $\alpha = 3$ and the EVecs are of the form $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \cdot t,\ t \in \mathbb{R}, t \neq 0$ (Exercise!). Therefore there is no eigenbasis. To find the general solution, we determine the solution y_1 as solution of a 2nd order ODE:

1. The 1st coordinate is $y_1' = 5y_1 + y_2$, the 2nd is $y_2' = -4y_1 + y_2$.
2. We calculate in the 1st coordinate
$$y_1' = 5y_1 + y_2 \implies y_2 \overset{*}{=} y_1' - 5y_1 \implies y_2' \overset{**}{=} y_1'' - 5y_1'.$$
3. Insert $*$ in the 2nd equation of the system: $y_2' = -4y_1 + \underbrace{y_1' - 5y_1}_{=y_2} = y_1' - 9y_1$.
4. With $**$ and $y_2' = y_1' - 9y_1$ in 3., we obtain a 2nd order ODE for y_1
$$y_1'' - 5y_1' = y_1' - 9y_1 \implies \boxed{y_1'' - 6y_1' + 9y_1 = 0}$$

How can we now find the general solution y_1 of this ODE? If we succeed in this, then also find y_2 with the equation $*$.

Definition. *Let $a, b \in \mathbb{R}$ and $g : x \mapsto g(x)$ be a function. An ODE of the form*

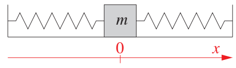
$$y''(x) + ay'(x) + by(x) = g(x)$$

is called a 2nd order linear ODE with constant coefficients. If $g = 0$, it is called homogeneous.

Note that again **no** products $y^2, (y')^2, y' \cdot y'', ...$ appear.

If possible try to reorder to get the form $y''(x) + ay'(x) + by(x) = g(x)$. The example ODE $5y''(x) - y'(x) = y(x) + \cos(x)$ is indeed the nonhomogeneous ODE $y''(x) - \frac{1}{5}y'(x) - \frac{1}{5}y(x) = \frac{1}{5}\cos(x)$.

Example harmonic oscillator 2nd order ODE play an important role in the application, especially with oscillations. Let's look at the example of the harmonic oscillator. A mass m moves on a rail and is connected to a spring on the left and right:



Let $x(t)$ be the position of the mass at time t with $x(0) = R$.

Without friction According to Newton's 2nd law (force = mass times acceleration) and Hooke's law of springs (repulsive spring force = spring constant k times deflection) applies $mx''(t) = -kx(t) \implies \boxed{x''(t) + \frac{k}{m}x(t) = 0.}$
The equation is often specified with the natural frequency ω

$$\boxed{x''(t) + \omega^2 x(t) = 0} \quad \text{with } \omega = \sqrt{\frac{k}{m}}.$$

With friction If we obtain a damped motion with

$$mx''(t) = -kx(t) \underbrace{-r}_{\text{friction}} x'(t) \implies \boxed{x''(t) + \frac{r}{m}x'(t) + \frac{k}{m}x(t) = 0.}$$

Solution in the homogeneous case With the approach $y(x) = e^{\lambda x}$ we obtain the characteristic equation equation $\lambda^2 + a\lambda + b = 0$ of a homogeneous differential equation $y''(x) + ay'(x) + by(x) = 0$. This quadratic equation has the solutions $\lambda_{1/2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

How do the three cases $a^2 - 4b > 0$, < 0 or $= 0$.

1. $a^2 - 4b > 0$, $\lambda_1 \neq \lambda_2$, two real solutions:

   General solution $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

2. $a^2 - 4b = 0$, $\lambda_1 = \lambda_2 = \alpha$, a real solution:

   General solution $y(x) = C_1 e^{\alpha x} + C_2 \underset{\uparrow}{x} e^{\alpha x} = (C_1 + C_2 x)e^{\alpha x}$.

3. $a^2 - 4b < 0$, conjugates complex solutions $\lambda_{1/2} = \alpha \pm i\beta$:

   General solution $y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$.

Exercises

1. Compare this with the three cases for determining the general solution of a linear 2×2-system $y' = Ay$.

2. Solve and plot the solution

   (a) For the Harmonic Oszillator (both cases)
   (b) $y'' - y' - 2y = 0$.
   (c) $y'' - 2\sqrt{3}y' + 2y = 0$
   (d) $y''(x) + 4y'(x) + 4y(x) = 0$ where $y(0) = 2, y'(0) = 4$
   (e) $x'' + 2x' + x = 0$ where $x(0) = x'(0) = 1$.

Application to Harmonic oscillator In the case **without friction** we are using the above recipe for the ODE $x''(t) + \frac{k}{m}x(t) = 0$. With $\omega^2 = \frac{k}{m}$ the characteristic equation is $\lambda^2 + \omega^2 = 0$ with solutions $\lambda_{1,2} = \pm\omega\, i$. We are in the third case above and the solution is $x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$.

Exercise Determine the constants with initial values $x(0) = d, x'(0) = 0$.

Types of damping For the Harmonic oscillator **with friction** we investigate the ODE $x''(t)+\frac{r}{m}x'(t)+\frac{k}{m}x(t) = 0$. Again set $\omega^2 = \frac{k}{m}$ and also $2\mu = \frac{r}{m}$. The characteristic equation is $\lambda^2+2\mu+\omega^2 = 0$ with solutions $\lambda_{1,2} = -\mu\pm\sqrt{\mu^2 - \omega^2}$. We distinguish strong damping in case $\mu > \omega \implies \lambda_1 \neq \lambda_2 < 0$ real solutions; critical damping if $\mu = \omega \implies \alpha = \lambda_1 = \lambda_2 < 0$ a real solution and low damping in case of $\mu < \omega \implies \lambda_{1,2} = -\mu \pm i\beta$ complex solutions.

Solution in the nonhomogeneous case Let $y''(x) + ay'(x) + by(x) = g(x)$ be nonhomogeneous. As in the 1st order case we have the following: Let y_H be the **general solution of the homogeneous ODE** $y'' + ay' + by = 0$ and let y_{sp} **be a** special solution of $y'' + ay' + by = g$. Then $\boxed{y = y_H + y_{sp}}$ is the **general solution of the nonhomogeneous ODE** $y''+ay'+by = g$. Finding such a special solution y_{sp} requires an investigation or guessing or an approach that depends on g.

Solving a 2×2-ODE system as a 2nd order ODE To find the solution of the system $y' = Ay$ with $A = \left(\begin{smallmatrix} 5 & 1 \\ -4 & 1 \end{smallmatrix}\right)$ we derived the ODE $\boxed{y_1'' - 6y_1' + 9y_1 = 0}$ with characteristic equation $\lambda^2 - 6\lambda + 9 = 0 = (\lambda - 3)^2$. Note that this is also the characteristic polynomial $p_A(\lambda)$ of the matrix A. Thus, the general solution for the 1st coordinate is $\boxed{y_1(x) = (C_1 + C_2x)e^{3x}}$ according to above section. With the equations in the ODE system we got $y_2 \overset{*}{=} y_1' - 5y_1$. Plugging in y_1 and y_1' it follows:

$$y_2 \overset{*}{=} y_1' - 5y_1 = \left((C_1 + C_2x)e^{3x}\right)' - 5\left((C_1 + C_2x)e^{3x}\right) = -2C_1e^{3x} + C_2(1 - 2x)e^{3x}$$

Exercise Check the last step by differentiating and sorting)

Collecting the two solution we get the general solution of the system:

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} C_1e^{3x} + C_2x\, e^{3x} \\ -2C_1e^{3x} + C_2(1 - 2x)e^{3x} \end{pmatrix} = \begin{pmatrix} C_1e^{3x} \\ -2C_1e^{3x} \end{pmatrix} + \begin{pmatrix} C_2x\, e^{3x} \\ C_2(1 - 2x)e^{3x} \end{pmatrix}$$
$$= C_1e^{3x} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\, e^{3x} \begin{pmatrix} x \\ 1 - 2x \end{pmatrix}$$

The general case Let us verify that we can also generally apply the solution of a system $y' = Ay$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can lead back to the solution of a 2nd order ODE.

Let $b \neq 0 \neq c$. Otherwise the two differential equations are decoupled.

1. Multiplying the first coordinate of the system $y_1' = ay_1 + by_2$ by d gives
$$dy_1' = ady_1 + bdy_2 \implies dy_1' - ady_1 = bdy_2.$$
By differentiating we obtain $y_1'' = ay_1' + by_2' \implies y_1'' - ay_1' = by_2'$.
2. The second coordinate $y_2' \overset{*}{=} cy_1 + dy_2$ multiplied by b and 1. eliminate y_2
$$y_1'' - ay_1' = by_2' \overset{*}{=} bcy_1 + bdy_2 = bcy_1 + dy_1' - ady_1.$$
3. We sort $y_1'' - (a+d)y_1' + (ad - bc)y_1 = 0$. Alternative notation with the trace $\mathrm{tr}(A) = a + d$, the sum of the diagonal entries: $\boxed{y_1'' - \mathrm{tr}(A)y_1' + \det(A)y_1 = 0}$

The characteristic equation of this ODE is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$ and is equal to the characteristic polynomial of the matrix A in which the trace $\mathrm{tr}(A)$ is the sum of the EVal and $\det(A)$ is the product.

On the other hand, if we have an ODE $y''(x) + ay'(x) + by(x) = 0$, how does the associated system look like $y' = Ay$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose solution corresponds to the solution of the 2nd order ODE? We get $\left\{ \begin{array}{rcr} y_1'(x) & = & y_2(x) \\ y_2'(x) & = & -by_1(x) - ay_2(x) \end{array} \right\}$ with $A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$ and this corresponds to $y_1''(x) + ay_1'(x) + by_1(x) = 0$.

Example Harmonic oscillator with $x''(t) + \omega^2 x(t) = 0$ and $\omega = \sqrt{\frac{k}{m}}$ leads to the matrix $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$.

Exercises Translate the 2nd order ODEs in the above exercise into a 2×2-system:

1. $y'' - y' - 2y = 0$
2. $y'' - 2\sqrt{3}y' + 2y = 0$
3. $y''(x) + 4y'(x) + 4y(x) = 0$ where $y(0) = 2, y'(0) = 4$
4. $x'' + 2x' + x = 0$ where $x(0) = x'(0) = 1$.

Plot the solutions of the system as curves in the plane and apply the classification using EVal.

Comparison of the methods If we are faced with a matrix A that does not allow an eigenbasis we can apply the matrix exponential to solve a homogeneous $n \times n$ system $y' = Ay$. In the case $n = 2$ for a 2×2 matrix, there is also a variant via a 2nd order ODE. Why do these two provide the same solution?

We will show this with the example $A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$ with double EW $\alpha = 3$.

The associated EVecs are of the form $t\begin{pmatrix}1\\-2\end{pmatrix}$ with $t \in \mathbb{R}, t \neq 0$. There is no eigenbasis, so we cannot directly find a basis of the solution space $\mathcal{L}_A$.

The computer calculates $e^{xA} = \begin{pmatrix} e^{3x}(1+2x) & e^{3x}x \\ -4e^{3x}x & e^{3x}(1-2x) \end{pmatrix} = (b_1(x)\ b_2(x))$. The general solution is therefore

$$y(x) = C_1 b_1(x) + C_2 b_2(x) = C_1 e^{3x}\begin{pmatrix}1+2x\\-4x\end{pmatrix} + C_2\, e^{3x}\begin{pmatrix}x\\1-2x\end{pmatrix}$$
$$= C_1 e^{3x}\left(x\begin{pmatrix}2\\-4\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\right) + C_2\, e^{3x}\begin{pmatrix}x\\1-2x\end{pmatrix}.$$

Without the matrix exponential, we can also determine the general solution using a 2nd order ODE. In this case it is $\boxed{y_1'' - 6y_1' + 9y_1 = 0}$ with solution

$$y(x) = C_1 e^{3x}\begin{pmatrix}1\\-2\end{pmatrix} + C_2\, e^{3x}\begin{pmatrix}x\\1-2x\end{pmatrix}.$$

These two representations of the general solution $y(x)$ look different at first. If we choose an initial vector, for example $y(0) = y_0 = \begin{pmatrix}1\\1\end{pmatrix}$, this results in the same solution function in each case. We calculate:

$$y(x) = e^{xA}y_0 = \begin{pmatrix} e^{3x}(1+2x) & e^{3x}x \\ -4e^{3x}x & e^{3x}(1-2x) \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} e^{3x} + 3e^{3x}x \\ e^{3x} - 3e^{3x}x \end{pmatrix}.$$

In the other case, we determine the two constants we are looking for C_1 and C_2 by inserting $x = 0$ into $y(x) = C_1 e^{3x}\begin{pmatrix}1\\-2\end{pmatrix} + C_2\, e^{3x}\begin{pmatrix}x\\1-2x\end{pmatrix}$.

The system of linear equations $y(0) = C_1\begin{pmatrix}1\\-2\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ then gives $C_1 = 1$ and $C_2 = 3$. With these, the result is again $y(x) = \begin{pmatrix} e^{3x} + 3e^{3x}x \\ e^{3x} - 3e^{3x}x \end{pmatrix}$.