Probability and Statistics

Exercise sheet 1

MC 1.1. Let $A, B \subseteq \Omega$. Which of the following does **not** hold? (Exactly one answer is correct.)

(a) $(A \setminus B)^c = B \cup A^c$.

(b) $(A \cup B)^c = A^c \cap B^c$.

(c) $A \setminus B^c = A \cap B$.

(d) $(A \cup B)^c = A^c \cup B^c$.

MC 1.2. Let $\Omega := \{\omega_1, \omega_2, \omega_3\}$. Which of the following does **not** define a σ-algebra on Ω? (Exactly one answer is correct.)

(a) $\mathcal{F}_1 := \big\{\emptyset, \Omega\big\}$.

(b) $\mathcal{F}_2 := \big\{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\big\}$.

(c) $\mathcal{F}_3 := \big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\big\}$.

(d) $\mathcal{F}_4 := \big\{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\big\}$.

MC 1.3. Let $\Omega := \{0, 1\}$, and $\mathcal{F} := 2^\Omega$. Which of the following define a probability measure on Ω? (The number of correct answers is between 0 and 4.)

(a) $\mathbb{P}[\emptyset] = \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = \mathbb{P}[\{0, 1\}] = \frac{1}{4}$.

(b) $\mathbb{P}[\emptyset] = \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = 0$ and $\mathbb{P}[\{0, 1\}] = 1$.

(c) $\mathbb{P}[\emptyset] = 0, \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = \frac{1}{2}$, and $\mathbb{P}[\{0, 1\}] = 1$.

(d) $\mathbb{P}[\emptyset] = 0, \mathbb{P}[\{0\}] = \frac{1}{4}, \mathbb{P}[\{1\}] = \frac{1}{2}$, and $\mathbb{P}[\{0, 1\}] = \frac{3}{4}$.

Exercise 1.4. [Settlers of Catan] We are playing the board game Settlers of Catan. The game board consists of landscapes that are labeled with integers between 2 and 6 or between 8 and 12. In each round, two dice are rolled and those landscapes whose number matches the sum of the dice rolls yield resources.

(a) Define the sample space $\Omega := \{(w_1, w_2) \mid w_1, w_2 \in \{1, 2, 3, 4, 5, 6\}\}$. Identify the event {the landscapes with number 9 yield resources} as a subset of Ω.

(b) Which landscapes (i.e., which numbers) are expected to yield resources most frequently and least frequently? Why?

(c) A player has a choice: Either they receive future resources from a landscape with number 8 or from both landscapes with number 4 and 12. What should they choose and why? (We assume that the type of resource does not influence the decision.)

Exercise 1.5. [Biased coins] We assume that we have two biased coins, gold and silver, in an urn. The probability that the gold coin lands on heads is $p_g \in (0,1)$, and for silver, it is $p_s \in (0,1)$. In each trial, a coin is drawn from the urn, tossed, and then returned to the urn. We conduct the random experiment twice.

(a) Specify an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (We assume that the gold and silver coin are each drawn with probability 1/2.)

(b) Which element of $\mathcal{F}$ corresponds to the event $A = \{\text{The first coin drawn is silver}\}$.

(c) Which element of $\mathcal{F}$ corresponds to the event $B = \{\text{Heads is obtained twice}\}$.

(d) Compute $\mathbb{P}[A]$, $\mathbb{P}[B]$, and $\mathbb{P}[A \cap B]$.

Exercise 1.6. [Properties of a σ-algebra]

(a) [De Morgan's Law] Let $(A_i)_{i \geq 1}$ be a sequence of arbitrary sets. Show that the following holds:

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} (A_i)^c.$$

Let $\mathcal{F}$ be a σ-algebra on Ω.

(b) Show that $\emptyset \in \mathcal{F}$.

(c) Let $(A_i)_{i \geq 1}$ be a sequence of events, i.e., $A_i \in \mathcal{F}$ for all $i \geq 1$. Show that

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

(d) Let $A, B \in \mathcal{F}$. Show that $A \cup B \in \mathcal{F}$.

(e) Let $A, B \in \mathcal{F}$. Show that $A \cap B \in \mathcal{F}$.

Exercise 1.7. [Properties of a probability measure] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Show that $\mathbb{P}[\emptyset] = 0$.

(b) Let $k \geq 1$, and let $A_1, \ldots, A_k$ be k pairwise disjoint events. Show that

$$\mathbb{P}[A_1 \cup \cdots \cup A_k] = \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_k].$$

(c) Let A be an event. Show that $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$.

(d) Let A and B be two arbitrary events (not necessarily disjoint). Show that the addition rule

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

holds.