Probability and Statistics

Exercise sheet 12

MC 12.1. Let $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(-1, \sigma^2)$, where $\sigma^2 > 0$ is unknown. What is the value of σ^2 if $\mathbb{P}[X \leq -1] = \mathbb{P}[Y \geq 2]$? (Exactly one answer is correct.)

(a) $\sigma^2 = 9$.

(b) $\sigma^2 = 1$.

(c) $\sigma^2 = 4$.

(d) $\sigma^2 = 2$.

MC 12.2. Which of the following statements about statistical tests are true? (The number of correct answers is between 0 and 4.)

(a) If the null hypothesis is not rejected, we conclude that it must be true.

(b) The statistical test measures the probability that the null hypothesis is true.

(c) It is possible that a test rejects the null hypothesis even though it is true. However, we control the probability of this event.

(d) The result of the statistical test is random.

Exercise 12.3. We suspect that the consumption of sodium-rich foods has certain effects on blood pressure. Therefore, we conduct a study in which we first measure the blood pressure of 1000 individuals. These individuals then adopt a diet that is very high in sodium. After doing so, we measure their blood pressure again. Let $X_1, \ldots, X_{1000}$ denote the random variables representing the differences in blood pressure values (after minus before). We assume that the X_i are independent with $X_i \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$ is known, but $\mu \in \mathbb{R}$ is unknown. Design a test to determine whether sodium has an effect on blood pressure.

(a) Formulate the null hypothesis and the alternative.

(b) Find a test statistic and the critical region at the 5% level.

(c) Suppose that $\sigma^2 = 1$ and $\sum_{i=1}^{1000} x_i = 80.2$. What is the result of the test?

Exercise 12.4. A pharmaceutical company is introducing a new drug and wants to conduct a study to examine whether the effectiveness of this drug exceeds 60%. To do so, they administer the drug to 1000 individuals and collect the data. For simplicity, we assume that the drug either worked or did not work for each person.

(a) Find a suitable class of distributions for the random sample $X_1, \ldots, X_{1000}$ and formulate the null and alternative hypotheses to test whether the effectiveness exceeds 60%.

(b) Consider the test statistic $S := \sum_{i=1}^{1000} X_i$. Use an appropriate approximation for the distribution of S under the null hypothesis.

(c) Find the approximate critical region at the significance level $\alpha = 0.05$.

(d) In our study, the drug was effective for 650 individuals. What is the result of this test?

Exercise 12.5. Assume that $X_1, \dots, X_n$, $n \in \mathbb{N}$, are i.i.d. random variables with $\mathbb{E}[X_1^2] < \infty$. Let $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathrm{Var}[X_1]$. Suggest suitable (exact or approximate) tests at level $\alpha \in (0,1)$ for the following situations.

(a) Test $\mathrm{H}_0 : \mu = 0$ against $\mathrm{H}_1 : \mu \neq 0$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the variance σ^2 is **known**.

(b) Test $\mathrm{H}_0 : \mu = 0$ against $\mathrm{H}_1 : \mu \neq 0$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the variance σ^2 is **unknown**.

(c) Test $\mathrm{H}_0 : \sigma^2 = 1$ against $\mathrm{H}_1 : \sigma^2 > 1$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the mean $\mu \in \mathbb{R}$ is **unknown**.

(d) Test $\mathrm{H}_0 : \mu = 1$ against $\mathrm{H}_1 : \mu < 1$, assuming that the variance σ^2 is **known**, but the distribution of X_i is **unknown**.

(e) Test $\mathrm{H}_0 : \mu = 1$ against $\mathrm{H}_1 : \mu < 1$, assuming that both the variance σ^2 and the distribution of X_i are **unknown**.

Exercise 12.6. A six-sided die is to be tested for whether it is loaded and more likely to land on six. For this purpose, an experiment is conducted in which the die is rolled ten times and the result of each roll is recorded. We assume all rolls are independent and that the probability of rolling a 1, 2, 3, 4, or 5 is the same. We model the outcomes of the rolls as a sample $X_1, \dots, X_{10}$, where $X_i = 1$ indicates that the i-th roll was a six, and $X_i = 0$ otherwise. We obtain the following results:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0	0	1	0	1	1	0	1	0	0

(a) Determine a suitable model $(\mathbb{P}_\theta)_{\theta \in \Theta}$, i.e., a parameter space and the distributions of $X_1, \dots, X_{10}$ under each $\mathbb{P}_\theta$.

(b) Formulate a suitable null hypothesis H_0 and alternative hypothesis H_1.

(c) Let $T = \sum_{i=1}^{10} X_i$ be the test statistic. What distribution does T follow?

(d) Let $K = (4, 10]$ be the rejection region. Compute the probability of a Type I error.

(e) Describe the test decision based on the observed results.

Exercise 12.7. Let $X_1, \dots, X_{12}$ be independent and identically distributed as $\mathcal{N}(\mu, \sigma^2)$ under $\mathbb{P}_\theta$, where $\theta = \mu$ is an unknown parameter. The standard deviation $\sigma = 0.0499$ is known. The following sample data are given:

x_1	x_2	x_3	x_4	x_5	x_6
1.00781	1.00646	1.00801	1.00833	1.00738	1.00687

x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
1.00783	1.00936	1.00564	1.00543	1.00794	1.01060

We test the hypothesis $\mathrm{H}_0 \colon \mu = \mu_0$ against the alternative $\mathrm{H}_1 \colon \mu \neq \mu_0$, where $\mu_0 = 1.0085$.

(a) Determine a and b such that the test statistic $T := \frac{1}{a}(\sum_{i=1}^{12} X_i + b)$ follows $\mathcal{N}(0,1)$ under $\mathbb{P}_{\mu_0}$.

(b) Let $K := (-\infty, -c) \cup (c, \infty)$ be the rejection region for some $c \geq 0$. Test H_0 against H_1 at the 5% significance level.

(c) Compute the power of the test at $\mu = 1.008$.

Quantile table for the standard normal distribution

0.5	0.75	0.9	0.95	0.975	0.99	0.995	0.999
0	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

For instance, $\Phi^{-1}(0.9) = 1.2816$, where Φ is the distribution function of $\mathcal{N}(0,1)$.

Table of standard normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

For instance, $\mathbb{P}[Z \leq 1.96] = 0.975$.