Probability and Statistics

Exercise sheet 12 - Solutions

MC 12.1. Let $X \sim \mathcal{N}(1,4)$ and $Y \sim \mathcal{N}(-1,\sigma^2)$, where $\sigma^2 > 0$ is unknown. What is the value of σ^2 if $\mathbb{P}[X \leq -1] = \mathbb{P}[Y \geq 2]$? (Exactly one answer is correct.)

(a) $\sigma^2 = 9$.

(b) $\sigma^2 = 1$.

(c) $\sigma^2 = 4$.

(d) $\sigma^2 = 2$.

Solution: (a) is correct. Due to the symmetry of the normal distribution, we have

$$\mathbb{P}[X \leq -1] = \mathbb{P}\left[\frac{X-1}{\sqrt{4}} \leq \frac{-1-1}{\sqrt{4}}\right] = \Phi(-1) = 1 - \Phi(1),$$
$$\mathbb{P}[Y \geq 2] = 1 - \mathbb{P}[Y \leq 2] = 1 - \mathbb{P}\left[\frac{Y+1}{\sigma} \leq \frac{2+1}{\sigma}\right] = 1 - \Phi\left(\frac{3}{\sigma}\right).$$

Furthermore,

$$1 - \Phi(1) = 1 - \Phi\left(\frac{3}{\sigma}\right) \iff 1 = \frac{3}{\sigma} \iff \sigma = 3 \iff \sigma^2 = 9.$$

MC 12.2. Which of the following statements about statistical tests are true? (The number of correct answers is between 0 and 4.)

(a) If the null hypothesis is not rejected, we conclude that it must be true.

(b) The statistical test measures the probability that the null hypothesis is true.

(c) It is possible that a test rejects the null hypothesis even though it is true. However, we control the probability of this event.

(d) The result of the statistical test is random.

Solution:

(a) is not true.

(b) is also not true. We cannot assign a probability to the event {"the null hypothesis is true"}. This is because the truth of the hypothesis is unknown, but not random.

(c) is true.

(d) is also true. Since the observed data are random, the result of the statistical test is also random.

Exercise 12.3. We suspect that the consumption of sodium-rich foods has certain effects on blood pressure. Therefore, we conduct a study in which we first measure the blood pressure of 1000 individuals. These individuals then adopt a diet that is very high in sodium. After doing so, we measure their blood pressure again. Let $X_1, \dots, X_{1000}$ denote the random variables representing the differences in blood pressure values (after minus before). We assume that the X_i are independent with $X_i \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$ is known, but $\mu \in \mathbb{R}$ is unknown. Design a test to determine whether sodium has an effect on blood pressure.

(a) Formulate the null hypothesis and the alternative.

(b) Find a test statistic and the critical region at the 5% level.

(c) Suppose that $\sigma^2 = 1$ and $\sum_{i=1}^{1000} x_i = 80.2$. What is the result of the test?

Solution:

(a) We formulate the hypotheses as follows:

$$\mathrm{H}_0 : \mu = 0 \quad \text{and} \quad \mathrm{H}_1 : \mu \neq 0.$$

(b) We know that under the null hypothesis, the random variable $S := \sum_{i=1}^{1000} X_i$ follows the distribution $\mathcal{N}(0, 1000\sigma^2)$. Therefore, under the null hypothesis:

$$\frac{S}{\sqrt{1000\sigma^2}} \sim \mathcal{N}(0,1).$$

We want to find c such that

$$\mathbb{P}_0\left[|S| \geq c\right] = \mathbb{P}_0\left[S \notin (-c, c)\right] = 0.05.$$

We have

$$\mathbb{P}_0\left[S \notin (-c, c)\right] = 0.05 \iff \mathbb{P}_0\left[\frac{S}{\sqrt{1000\sigma^2}} \notin \left(-\frac{c}{\sqrt{1000\sigma^2}}, \frac{c}{\sqrt{1000\sigma^2}}\right)\right] = 0.05.$$

Due to the symmetry of the normal distribution, we further have

$$\begin{aligned}\mathbb{P}_0\left[\frac{S}{\sqrt{1000\sigma^2}} \notin \left(-\frac{c}{\sqrt{1000\sigma^2}}, \frac{c}{\sqrt{1000\sigma^2}}\right)\right] &= \Phi\left(-\frac{c}{\sqrt{1000\sigma^2}}\right) + \left(1 - \Phi\left(\frac{c}{\sqrt{1000\sigma^2}}\right)\right) \\ &= 2\left(1 - \Phi\left(\frac{c}{\sqrt{1000\sigma^2}}\right)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}_0\left[S \notin [-c, c]\right] = 0.05 &\iff 2\left(1 - \Phi\left(\frac{c}{\sqrt{1000\sigma^2}}\right)\right) = 0.05 \\ &\iff \Phi\left(\frac{c}{\sqrt{1000\sigma^2}}\right) = 0.975 \\ &\iff \frac{c}{\sqrt{1000\sigma^2}} = \Phi^{-1}(0.975) \\ &\iff \frac{c}{\sqrt{1000\sigma^2}} = 1.96 \\ &\iff c = 1.96\sqrt{1000\sigma^2}.\end{aligned}$$

The critical region is therefore $(-\infty, -1.96\sqrt{1000\sigma^2}] \cup [1.96\sqrt{1000\sigma^2}, \infty)$, and we reject the null hypothesis when $\sum_{i=1}^{1000} x_i \leq -1.96\sqrt{1000\sigma^2}$ or $\sum_{i=1}^{1000} x_i \geq 1.96\sqrt{1000\sigma^2}$.

(c) We have $\sum_{i=1}^{1000} x_i = 80.2 > 1.96\sqrt{1000} \approx 61.98$. Therefore, we reject the null hypothesis and conclude that a sodium-rich diet has a significant effect on blood pressure. We also observe that blood pressure increased. Hence, we can claim that sodium tends to increase blood pressure.

Exercise 12.4. A pharmaceutical company is introducing a new drug and wants to conduct a study to examine whether the effectiveness of this drug exceeds 60%. To do so, they administer the drug to 1000 individuals and collect the data. For simplicity, we assume that the drug either worked or did not work for each person.

(a) Find a suitable class of distributions for the random sample $X_1, \dots, X_{1000}$ and formulate the null and alternative hypotheses to test whether the effectiveness exceeds 60%.

(b) Consider the test statistic $S \coloneqq \sum_{i=1}^{1000} X_i$. Use an appropriate approximation for the distribution of S under the null hypothesis.

(c) Find the approximate critical region at the significance level $\alpha = 0.05$.

(d) In our study, the drug was effective for 650 individuals. What is the result of this test?

Solution:

(a) We model this situation by assuming that the X_i are independent and follow a Bernoulli distribution with parameter $p \in [0,1]$. We want to test whether $p > 0.6$. Therefore, we formulate the hypotheses as follows:

$$\mathrm{H}_0 : p \leq 0.6, \quad \text{and} \quad \mathrm{H}_1 : p > 0.6.$$

(b) We know that if the X_i, $i \in \{1, \dots, 100\}$, are independent Bernoulli variables with parameter $p \in [0,1]$, then $S \sim \mathrm{Binom}(1000, p)$. Under the null hypothesis, we assume $p \leq 0.6$. Since $p_0 = 0.6$ is the closest value to the alternative, we choose $S \sim \mathrm{Binom}(1000, 0.6)$. See (1) for more details.

We can use the central limit theorem. With $\mathbb{E}_{0.6}[X_i] = p_0 = 0.6$ and $\mathrm{Var}_{0.6}[X_i] = p_0(1-p_0) = 0.24$, we have approximately under the null hypothesis:

$$\frac{S - 1000 \times 0.6}{\sqrt{1000 \times 0.24}} \sim \mathcal{N}(0,1).$$

(c) We want to find c such that

$$\mathbb{P}_{0.6}[S \geq c] = 0.05.$$

Note that

$$\mathbb{P}_p[S \geq c] \leq \mathbb{P}_{0.6}[S \geq c] = 0.05 \tag{1}$$

for every $p \in [0, 0.6]$. In other words, the test has the correct significance level even if the true value of p is less than 0.6.

We have

$$\mathbb{P}_{0.6}[S \geq c] = \mathbb{P}_{0.6}\left[\frac{S-600}{\sqrt{240}} \geq \frac{c-600}{\sqrt{240}}\right] \approx 1 - \Phi\left(\frac{c-600}{\sqrt{240}}\right).$$

Moreover

$$1 - \Phi\left(\frac{c-600}{\sqrt{240}}\right) = 0.05 \iff 0.95 = \Phi\left(\frac{c-600}{\sqrt{240}}\right) \iff \Phi^{-1}(0.95) = \frac{c-600}{\sqrt{240}} \iff 1.6449 = \frac{c-600}{\sqrt{240}} \iff c = 1.6449\sqrt{240} + 600.$$

Since $1.6449\sqrt{240} + 600 \approx 625.5$, we obtain an approximate critical region $[626, \infty)$.

For comparison: the exact 0.95-quantile of $\mathrm{Binom}(1000, 0.6)$ is 625, so the approximation is very accurate.

(d) We see that $s = 650$ lies in the approximate critical region. Therefore, we reject the null hypothesis and are confident that the effectiveness exceeds 60%.

Exercise 12.5. Assume that $X_1, \ldots, X_n$, $n \in \mathbb{N}$, are i.i.d. random variables with $\mathbb{E}[X_1^2] < \infty$. Let $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathrm{Var}[X_1]$. Suggest suitable (exact or approximate) tests at level $\alpha \in (0,1)$ for the following situations.

(a) Test $\mathrm{H}_0 : \mu = 0$ against $\mathrm{H}_1 : \mu \neq 0$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the variance σ^2 is **known**.

(b) Test $\mathrm{H}_0 : \mu = 0$ against $\mathrm{H}_1 : \mu \neq 0$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the variance σ^2 is **unknown**.

(c) Test $\mathrm{H}_0 : \sigma^2 = 1$ against $\mathrm{H}_1 : \sigma^2 > 1$, assuming that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the mean $\mu \in \mathbb{R}$ is **unknown**.

(d) Test $\mathrm{H}_0 : \mu = 1$ against $\mathrm{H}_1 : \mu < 1$, assuming that the variance σ^2 is **known**, but the distribution of X_i is **unknown**.

(e) Test $\mathrm{H}_0 : \mu = 1$ against $\mathrm{H}_1 : \mu < 1$, assuming that both the variance σ^2 and the distribution of X_i are **unknown**.

Solution: We only provide the summaries for each test for the sake of brevity. The detailed steps of each test can be carried out analogously to the other exercises.

(a) Under H_0, the statistic

$$T_{(\mathrm{a})} := \frac{\overline{X}_n}{\sqrt{\frac{\sigma^2}{n}}}$$

follows the standard normal distribution. Since the alternative is two-sided, we reject the null hypothesis if

$$T_{(\mathrm{a})} < z_{\alpha/2} \quad \text{or} \quad T_{(\mathrm{a})} > z_{1-\alpha/2}, \quad \text{which is equivalent to} \quad |T_{(\mathrm{a})}| > z_{1-\alpha/2}.$$

(b) As the variance is unknown, we replace it with an estimator. Under H_0, the statistic

$$T_{(\mathrm{b})} \coloneqq \frac{\overline{X}_n}{\sqrt{\frac{S_n^2}{n}}}, \quad \text{where} \quad S_n^2 \coloneqq \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2,$$

follows the Student's t-distribution with $n-1$ degrees of freedom. Since the alternative is two-sided, we reject the null hypothesis if

$$T_{(\mathrm{b})} < t_{n-1,\alpha/2} \quad \text{or} \quad T_{(\mathrm{b})} > t_{n-1,1-\alpha/2}, \quad \text{which is equivalent to} \quad |T_{(\mathrm{b})}| > t_{n-1,1-\alpha/2},$$

where $t_{n-1,\alpha}$ denotes the α-quantile of the t-distribution with $n-1$ degrees of freedom.

(c) Under H_0, i.e., assuming $\sigma^2 = \sigma_0^2 \coloneqq 1$, the statistic

$$T_{(\mathrm{c})} \coloneqq \frac{(n-1)S_n^2}{\sigma_0^2} = (n-1)S_n^2$$

follows the χ^2_{n-1} distribution. Since the alternative is one-sided, we must be careful about which quantile to use. For large values of σ^2, both S_n^2 and the value of the statistic tend to be large, and so we reject the null hypothesis if

$$T_{(\mathrm{c})} > \chi^2_{n-1,1-\alpha},$$

where, as usual, $\chi^2_{n-1,\alpha}$ denotes the α-quantile of the χ^2_{n-1} distribution.

(d) As the distribution of the sample is unknown, we use the central limit theorem to construct an approximate test. Under H_0, the statistic

$$T_{(\mathrm{d})} \coloneqq \frac{\overline{X}_n - 1}{\sqrt{\frac{\sigma^2}{n}}}$$

approximately follows the standard normal distribution. Since the alternative is one-sided and the value of the test statistic tends to be low when μ is small, we reject the null hypothesis if

$$T_{(\mathrm{d})} < z_\alpha.$$

(e) As the variance is unknown, we once again replace it with an estimator. Under H_0, the statistic

$$T_{(\mathrm{e})} \coloneqq \frac{\overline{X}_n - 1}{\sqrt{\frac{S_n^2}{n}}}$$

still approximately follows the standard normal distribution. Since the alternative is one-sided and the value of the test statistic tends to be low when μ is small, we reject the null hypothesis if

$$T_{(\mathrm{e})} < z_\alpha.$$

Remark: It is clear that the test statistic $T_{(\mathrm{e})}$ could also be used in point (d). However, given our data, it is not clear what effect the estimate $s_n^2 = S_n^2(\omega)$ has on the value of the test statistic. For instance, if the estimate is higher than the true variance, it will make the value of the test statistic closer to zero, which may be undesirable since we are not testing the variance. Therefore, if we are truly confident that we know the variance (which is, however, usually not the case), it is better to use $T_{(\mathrm{d})}$.

Exercise 12.6. A six-sided die is to be tested for whether it is loaded and more likely to land on six. For this purpose, an experiment is conducted in which the die is rolled ten times and the result of each roll is recorded. We assume all rolls are independent and that the probability of rolling a 1, 2, 3, 4, or 5 is the same. We model the outcomes of the rolls as a sample $X_1, \ldots, X_{10}$, where $X_i = 1$ indicates that the i-th roll was a six, and $X_i = 0$ otherwise. We obtain the following results:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0	0	1	0	1	1	0	1	0	0

(a) Determine a suitable model $(\mathbb{P}_\theta)_{\theta\in\Theta}$, i.e., a parameter space and the distributions of $X_1, \ldots, X_{10}$ under each $\mathbb{P}_\theta$.

(b) Formulate a suitable null hypothesis H_0 and alternative hypothesis H_1.

(c) Let $T = \sum_{i=1}^{10} X_i$ be the test statistic. What distribution does T follow?

(d) Let $K = (4, 10]$ be the rejection region. Compute the probability of a Type I error.

(e) Describe the test decision based on the observed results.

Solution:

(a) From the assumptions, it follows that $X_1, \ldots, X_{10}$ are independent and each follows a Bernoulli distribution with success parameter $\theta \coloneqq \mathbb{P}_\theta[X_1 = 1] \in \Theta = [0, 1]$ under $\mathbb{P}_\theta$.

(b) Our null hypothesis is that the die is not loaded; i.e.,

$$\mathrm{H}_0 : \theta = \frac{1}{6}, \quad \text{i.e., } \Theta_0 = \left\{\frac{1}{6}\right\}.$$

The alternative hypothesis, that the die is loaded, is then

$$\mathrm{H}_1 : \theta > \frac{1}{6}, \quad \text{i.e., } \Theta_1 = \left(\frac{1}{6}, 1\right].$$

(c) Since $X_1, \ldots, X_{10}$ are independent and follow $\mathrm{Bernoulli}(\theta)$ under $\mathbb{P}_\theta$, the test statistic T follows a binomial distribution: $T \sim \mathrm{Binom}(10, \theta)$ under $\mathbb{P}_\theta$.

(d) The probability of a Type I error is

$$\begin{aligned}\mathbb{P}_{\frac{1}{6}}[T \in K] &= \mathbb{P}_{\frac{1}{6}}[T \geq 5] \\ &= \sum_{k=5}^{10} \binom{10}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{10-k} \\ &= \frac{1}{6^{10}} \left(\binom{10}{5} 5^5 + \binom{10}{6} 5^4 + \binom{10}{7} 5^3 \right. \\ &\qquad \left. + \binom{10}{8} 5^2 + \binom{10}{9} 5^1 + \binom{10}{10} 5^0 \right) \\ &= 0.0155.\end{aligned}$$

(e) Since $t(x_1, \ldots, x_{10}) = t(X_1(\omega), \ldots, X_{10}(\omega)) = T(\omega) = 4$ is not in the rejection region, we do not reject the null hypothesis.

Exercise 12.7. Let $X_1, \ldots, X_{12}$ be independent and identically distributed as $\mathcal{N}(\mu, \sigma^2)$ under $\mathbb{P}_\theta$, where $\theta = \mu$ is an unknown parameter. The standard deviation $\sigma = 0.0499$ is known. The following sample data are given:

x_1	x_2	x_3	x_4	x_5	x_6
1.00781	1.00646	1.00801	1.00833	1.00738	1.00687

x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
1.00783	1.00936	1.00564	1.00543	1.00794	1.01060

We test the hypothesis $\mathrm{H}_0 \colon \mu = \mu_0$ against the alternative $\mathrm{H}_1 \colon \mu \neq \mu_0$, where $\mu_0 = 1.0085$.

(a) Determine a and b such that the test statistic $T \coloneqq \frac{1}{a}(\sum_{i=1}^{12} X_i + b)$ follows $\mathcal{N}(0,1)$ under $\mathbb{P}_{\mu_0}$.

(b) Let $K \coloneqq (-\infty, -c) \cup (c, \infty)$ be the rejection region for some $c \geq 0$. Test H_0 against H_1 at the 5% significance level.

(c) Compute the power of the test at $\mu = 1.008$.

Solution:

(a) Since $\sum_{i=1}^{12} X_i \sim \mathcal{N}(12\mu_0, 12\sigma^2)$ under $\mathbb{P}_{\mu_0}$, it follows that

$$T = \frac{\sum_{i=1}^{12} X_i - 12\mu_0}{\sigma\sqrt{12}} \sim \mathcal{N}(0,1)$$

under $\mathbb{P}_{\mu_0}$. Therefore, we choose $a = \sigma\sqrt{12}$ and $b = -12\mu_0$.

(b) Let $\alpha = 0.05$. We perform a test with test statistic T and rejection region K, i.e., we reject the hypothesis if $|T| > c$ for some c to be determined. By definition of the significance level:

$$\begin{aligned}\alpha = \mathbb{P}_{\mu_0}[T \in K] = \mathbb{P}_{\mu_0}[T \notin [-c, c]] &= \mathbb{P}_{\mu_0}[T < -c] + \mathbb{P}_{\mu_0}[T > c] \\ &= \Phi(-c) + 1 - \Phi(c) = 2 - 2\Phi(c),\end{aligned}$$

since $T \sim \mathcal{N}(0,1)$ under $\mathbb{P}_{\mu_0}$. Thus,

$$\Phi(c) = 1 - \frac{\alpha}{2} = 1 - \frac{0.05}{2} = 0.975,$$

so $c = \Phi^{-1}(0.975) = z_{0.975} = 1.96$. The observed test statistic is

$$T(\omega) = t(x_1, \ldots, x_{12}) = -0.00598.$$

Thus, we do not reject the null hypothesis.

(c) The power of the test at point $\mu = 1.008$ is

$$\begin{aligned}
\mathbb{P}_\mu[T \in K] = \mathbb{P}_\mu[T \notin [-c, c]] \\
&= \mathbb{P}_\mu\left[\frac{\sum_{i=1}^{12} X_i - 12\mu}{\sigma\sqrt{12}} < -c + \frac{\sqrt{12}}{\sigma}(\mu_0 - \mu)\right] \\
&\quad + \mathbb{P}_\mu\left[\frac{\sum_{i=1}^{12} X_i - 12\mu}{\sigma\sqrt{12}} > c + \frac{\sqrt{12}}{\sigma}(\mu_0 - \mu)\right] \\
&= \Phi\left(-c + \frac{\sqrt{12}}{\sigma}(\mu_0 - \mu)\right) + 1 - \Phi\left(c + \frac{\sqrt{12}}{\sigma}(\mu_0 - \mu)\right) \\
&= \Phi(-1.93) + 1 - \Phi(1.99) = 0.0501.
\end{aligned}$$

This is a low value, and the reason is that μ is very close to μ_0.

Quantile table for the standard normal distribution

0.5	0.75	0.9	0.95	0.975	0.99	0.995	0.999
0	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

For instance, $\Phi^{-1}(0.9) = 1.2816$, where Φ is the distribution function of $\mathcal{N}(0,1)$.

Table of standard normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

For instance, $\mathbb{P}[Z \leq 1.96] = 0.975$.